STABILITY OF BOUNDARY LAYERS FOR THE KELLER-SEGEL SYSTEM WITH SINGULAR SENSITIVITY IN THE HALF-PLANE

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ABSTRACT. Though the boundary layer formation in the chemotactic process has been observed in experiment (cf. [63]), the mathematical study on the boundary layer solutions of chemotaxis models is just in its infant stage. Apart from the sophisticated theoretical tools involved in the analysis, how to impose/derive physical boundary conditions is a state-of-the-art in studying the boundary layer problem of chemotaxis models. This paper will proceed with a previous work [24] in one dimension to establish the stability of boundary layer solutions of the Keller-Segel model with singular sensitivity in a two-dimensional space (half-plane). Compared to the one-dimensional boundary layer problem, there are many new issues arising from multi-dimensions such as possible Prandtl type degeneracy, curl-free preservation and well-posedness of large-data solutions. In this paper, we shall derive appropriate physical boundary conditions and gradually overcome these barriers and hence establish the stability of boundary layer solutions of the singular Keller-Segel system in the half-plane as the chemical diffusion rate vanishes. We hope that our results and methods can shed lights on the understanding of underlying mechanisms of the boundary layer patterns observed in the experiment for chemotaxis such as the work by Tuval *et al* [63], and open a new window in the theoretical study of chemotaxis models.

1. INTRODUCTION

Chemotaxis, the movement of an organism in response to a chemical stimulus, has been proved to be a significant mechanism accounting for abundant biological processes, such as aggregation of bacteria [48, 64], slime mould formation [21], fish pigmentation [51], tumor angiogenesis [3–5], primitive streak formation [52], blood vessel formation [14], wound healing [55]. As such, the mathematical works on modeling and analysis of chemotaxis has been greatly boosted in the past few decades. Mathematical modeling of chemotaxis dates to the pioneering works of Keller and Segel in [29] with linear sensitivity and in [28, 30] with logarithmic singular sensitivity. This paper is concerned with the following Keller-Segel (KS) system with logarithmic sensitivity:

$$\begin{cases} u_t = \nabla\cdot(D\nabla u - \chi \frac{u}{c}\nabla c), & (\vec{x},t)\in\Omega\times(0,\infty),\\ c_t = \varepsilon\Delta c - uc, \end{cases} \tag{1.1}$$

where $u(\vec{x},t)$ and $c(\vec{x},t)$ denote cell density and chemical (signal) concentration at position $\vec{x}$, time t and the spatial domain $\Omega = \mathbb{R}^2_+ = \{\vec{x} = (x,y)\in\mathbb{R}^2 \mid y>0\}$. $D>0$ and $\varepsilon\geq 0$ are cell and chemical diffusion coefficients, respectively, and $\chi>0$ is referred to as the chemotactic coefficient measuring the strength of the chemotactic sensitivity. System (1.1) is the KS model proposed in [30] with linear nutrient consumption, and later found more applications to model the boundary movement of chemotactic bacterial populations [49] and to describe the dynamical interactions between vascular endothelial cells (denoted by u), and signaling molecules vascular endothelial growth factor (denoted by c), in the initiation of tumor angiogenesis in [33]. Since the chemical diffusion ε has been assumed to be negligible (small) in all these works [28, 33, 49] due to both mathematical simplicity and biological insignificancy, an immediate relevant

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question is whether the dynamics of (1.1) has significant difference between $\varepsilon = 0$ and $\varepsilon > 0$ small. Specifically we want to elucidate whether the solutions of (1.1) with $\varepsilon > 0$ converge to those with $\varepsilon = 0$ as ε vanishes. While attempting this question, one has to face another challenging issue of (1.1): the singularity at $c = 0$. Fortunately this singularity can be salvaged by a Cole-Hopf type transformation (cf. [32, 42]):

$$\vec{v} = -\nabla \ln c = -\frac{\nabla c}{c}, \tag{1.2}$$

which transforms (1.1) into a non-singular system of conservation laws:

$$\begin{cases} u_t - \nabla \cdot (u\vec{v}) = \Delta u, \quad (\vec{x},t) \in \Omega \times (0,\infty), \\ \vec{v}_t + \nabla(\varepsilon|\vec{v}|^2 - u) = \varepsilon \Delta \vec{v}, \\ (u,\vec{v})(\vec{x},0) = (u_0,\vec{v}_0)(\vec{x}), \end{cases} \tag{1.3}$$

where we have appended initial data for completeness and taken $D = \chi = 1$ for brevity but our analysis in this paper directly carries to generic positive parameters D, $\chi > 0$.

Under the transformation (1.2), our question raised above boils down to investigate the vanishing diffusion limit of (1.3) as $\varepsilon \to 0$, which is an intriguing mathematical problem alone despite of its relevance to biology, since the vanishing advection needs to be considered along with vanishing diffusion due to the dual effect of ε. There has been several works investigating the vanishing diffusion limit of (1.3) as $\varepsilon \to 0$ in the literature. First in the whole space, it is shown that traveling wave solutions in $\mathbb{R}$ (cf. [43]) or global small-data solution of the Cauchy problem (cf. [53, 66]) in $\mathbb{R}^d$ $(d = 2,3)$ of (1.3) is uniformly convergent in ε, namely $(u^\varepsilon, \vec{v}^\varepsilon)$ converge to $(u^0, \vec{v}^0)$ in L^∞-norm as $\varepsilon \to 0$, where $(u^\varepsilon, \vec{v}^\varepsilon)$ denotes the solution of (1.3) with $\varepsilon \geq 0$. In a bounded interval $\Omega = (0,1)$, the solutions is still convergent (cf. [67]) in ε when (1.3) is endowed with the mixed homogeneous Neumann-Dirichlet boundary conditions

$$u_x|_{x=0,1} = v|_{x=0,1} = 0, \text{ for } \varepsilon \geq 0.$$

However if Dirichlet boundary conditions are prescribed, the situation is more complicated in that one can not preassign a boundary value for v^0 which is intrinsically determined by the second equation of (1.3) with $\varepsilon = 0$ as $v^0|_{x=0,1} = v_0|_{x=0,1} + \int_0^t u_x^0|_{x=0,1}\, d\tau$. Thus the appropriate Dirichlet boundary conditions should be imposed as (cf. [37]):

$$\begin{cases} u|_{x=0,1} = \bar{u} \geq 0, \quad v|_{x=0,1} = \bar{v} & \text{if } \varepsilon > 0, \\ u|_{x=0,1} = \bar{u} \geq 0 & \text{if } \varepsilon = 0, \end{cases} \tag{1.4}$$

where $\bar{u} \geq 0$, $\bar{v} \in \mathbb{R}$ are constants. Hence if the boundary value for v with $\varepsilon > 0$ does not match the one with $\varepsilon = 0$ determined by the second equation of (1.3), boundary layers for solution component v (i.e. rapid change of v near the boundary) will be present as ε is small. The above results imply that chemotaxis KS models with conventional Neumann (or zero-flux) boundary conditions will not generate boundary layers. To describe boundary layer phenomenon driven by chemotaxis observed in the experiment (e.g. [63]), Dirichlet boundary conditions are more relevant. Indeed boundary layer problem has been an important topic arising in the study of the inviscid limit of the Navier-Stokes equations near a boundary and has been one of the most fundamental issue in fluid mechanics attracting extensive studies (cf. [10, 12, 13, 26, 65, 70, 71]) since the pioneering work [56] by Prandtl in 1904. The existence of boundary layers for the transformed KS model (1.3) subject to Dirichlet boundary condition (1.4) has been numerically verified in [37] and rigorously proved in [25] in one dimension followed by a recent work [24] on the stability of boundary layers. This paper will proceed to investigate the boundary layer problem of (1.3) in two dimensions, which pertains to more realistic situations (cf. [63]). Due to the special structure of (1.3), there are several essential differences between one and two dimensions as to be detailed below. In two dimensions, $\vec{v}$ is a two-component vector from (1.2) and we denote $\vec{v} = (v_1, v_2)$ in the sequel. Then from the Cole-Hopf transformation (1.2), the curl for $\vec{v}$ must be intrinsically free:

$$\nabla \times \vec{v} = \partial_x v_2 - \partial_y v_1 = 0, \tag{1.5}$$

which implies that $\nabla|\vec{v}|^2 = 2\vec{v}\cdot\nabla\vec{v}$. Then the second equation of (1.3) becomes $\vec{v}_t + 2\varepsilon\vec{v}\cdot\nabla\vec{v} - \nabla u = \varepsilon\Delta\vec{v}$, which is surprisingly analogous to the incompressible Navier-Stokes (INS) equations by setting $\vec{w} = \vec{v}$ and $p = -u$:

$$\begin{cases} \vec{w}_t + \vec{w}\cdot\nabla\vec{w} + \nabla p = \varepsilon\Delta\vec{w}, & (\vec{x},t) \in \quad \Omega\times(0,\infty), \\ \nabla\cdot\vec{w} = 0, \end{cases} \tag{1.6}$$

where $\vec{w}$ is the fluid velocity and p the pressure. It is well-known that the inviscid limit of the INS equations will generate boundary layers if the following physical boundary conditions (e.g. see [8, 45]) are prescribed:

$$\begin{cases} \vec{w}|_{\partial\Omega} = 0 & \text{if } \varepsilon > 0, \\ \vec{w}\cdot\vec{n}|_{\partial\Omega} = 0 & \text{if } \varepsilon = 0, \end{cases}$$

where $\vec{n}$ is the unit outward normal vector of $\partial\Omega$. However, the convergence of solutions of the INS equations to its limiting Euler equations (namely (1.6) with $\varepsilon = 0$), in two or higher dimensions as $\varepsilon \to 0$ still remains unjustified due to the appearance of (degenerate) Prandtl's boundary layer equations (see [56]) whose well-posedness in Sobolev spaces is open except for analytic or monotonic data [1, 8, 15, 44, 50]. As such, due to the analogy between (1.3) and the INS equations, a natural concern is whether the KS system (1.3) with Dirichlet boundary conditions in two dimensions will generate similar Prandtl's boundary layers making the vanishing limit problem as $\varepsilon \to 0$ unverifiable? This question does not exist in one dimension but must be first elucidated in higher dimensions (see more details later) before taking the next step. Moreover the system (1.3) is invariant under the scaling for any $\lambda > 0$: $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t)$, $\vec{v}_\lambda(x,t) = \lambda\vec{v}(\lambda x, \lambda^2 t)$ which indicates that $d = 2$ is the critical space dimension of (1.3) in the framework of Sobolev spaces, and $d = 3$ is supercritical while $d = 1$ is subcritical, same as the Navier-Stokes equations (see [6]). But analysis of (1.3) is somewhat more difficult than the INS equations due to the lack of the divergence-free condition which is critical for the existence of large solutions to the INS equations in two dimensions (e.g. see [11, 46]). Indeed, although large-data solutions of (1.3) in one dimension have been obtained, none of the large-data solutions has been obtained in multi-dimensions so far even for the critical space dimension $d = 2$ (cf. [53, 66]). This is the second difference from the one-dimensional case. Thirdly, in order to preserve the curl-free condition (1.5) so that the results of (1.3) can be transferred to the original Keller-Segel system (1.1), the condition (1.5) has to be taken into account when prescribing boundary conditions. However no such concern is needed in one dimension.

Bearing these structural differences between one and two dimensions in mind, we shall exploit the zero-diffusion (inviscid) limit and boundary layers for the system (1.3) with Dirichlet boundary conditions in two dimensions in this paper. For simplicity, we consider the problem in the half plane $\Omega = \mathbb{R}^2_+ = \{\vec{x} = (x,y) \in \mathbb{R}^2 \mid y > 0\}$ and hence $\partial\Omega = \{(x,y) \in \mathbb{R}^2 | \, y = 0\}$. Taking the curl on both sides of the second equation of (1.3), one can get $\partial_t(\nabla\times\vec{v}) = \varepsilon\Delta(\nabla\times\vec{v})$. This indicates that to preserve the intrinsic curl-free condition (1.5), we ought to impose $\nabla\times\vec{v}_0 = 0$ along with the condition $\nabla\times\vec{v}|_{\partial\Omega} = (\partial_x v_2 - \partial_y v_1)|_{\partial\Omega} = 0$ for $\varepsilon > 0$. Therefore the boundary conditions of components v_1 and v_2 are dependent and the Dirichlet boundary conditions (for u and v_2) of (1.3) with $\varepsilon \geq 0$ are prescribed as:

$$\begin{cases} u|_{y=0} = \bar{u}(x,t), \ (\nabla\times\vec{v})|_{y=0} = 0, \ v_2|_{y=0} = \bar{v}(x,t) & \text{if } \varepsilon > 0, \\ u|_{y=0} = \bar{u}(x,t) & \text{if } \varepsilon = 0, \end{cases} \tag{1.7}$$

where $\bar{u}(x,t)$ and $\bar{v}(x,t)$ are functions of x and t and the component v_1 subjects to the Neumann boundary condition $\partial_y v_1|_{y=0} = \partial_x \bar{v}(x,t)$.

We shall study the stability of boundary layers for system (1.3) with (1.7) in the present paper. By the boundary layer theory [56, 61], we anticipate that the solution $(u^\varepsilon, \vec{v}^\varepsilon)$ of (1.3) with (1.7) with small $\varepsilon > 0$ consists of two parts: inner (boundary) layer profile and outer layer profile (the solution profile with $\varepsilon = 0$). Note that the thickness of boundary layers in one dimension has been formally justified as $O(\varepsilon^{1/2})$ in appendix of [24], which also holds for (1.3), (1.7) in two dimensions. Furthermore the inner boundary layer for u-component will be absent since the boundary conditions for u between $\varepsilon > 0$ and $\varepsilon = 0$ are consistent. By $(u^\varepsilon, \vec{v}^\varepsilon)$ and $(u^0, \vec{v}^0)$ we denote the solution of (1.3) with (1.7) with respect to $\varepsilon > 0$ and $\varepsilon = 0$, respectively. Then $(u^\varepsilon, \vec{v}^\varepsilon)$ is expected to have the following structure:

$$
\begin{aligned}
&u^\varepsilon(x,y,t) = u^0(x,y,t) + \mathscr{O}(\varepsilon^{1/2}),\\
&\vec{v}^\varepsilon(x,y,t) = \vec{v}^0(x,y,t) + \Big(v_1^{B,0}\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big), v_2^{B,0}\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big)\Big) + \mathscr{O}(\varepsilon^{1/2}),
\end{aligned}
\tag{1.8}
$$

where the outer layer profile $(u^0, \vec{v}^0) = (u^0, v_1^0, v_2^0)$ is the solution of (1.3) and (1.7) with $\varepsilon = 0$, and $(v_1^{B,0}, v_2^{B,0})$ denotes the inner layer profile which rapidly adjust from a value away from the boundary layer to another value on the boundary. If (1.8) holds, we say the boundary layer solution $(v^\varepsilon, \vec{v}^\varepsilon)$ is stable with respect to ε.

Due to various similarities between the second equation of (1.3) and the INS equations, justifying (1.8) seems to be a great challenge at first glance due to the possible presence of degenerate Prandtl type equation (as INS equations do) whose well-posedness with general initial data in Sobolev space still remains as a grand open question in spite of numerous attempts (cf. [15, 23, 59, 60, 68, 69]). However, thanks to the special structure of (1.3), the nonlinear trouble convection term $\varepsilon\nabla|\vec{v}|^2$ in (1.3) vanishes as $\varepsilon \to 0$ and the resulting limit equation $\vec{v}_t + \nabla u = 0$ is fundamentally different from the Euler equation - limit equation of INS. Indeed a formal asymptotic analysis will show that the boundary layer equations are not of Prandtl's type in two dimensions (see details in section 2). This key observation promises us a possibility to justifying (1.8), although we foresee that the appearance of ε in front of the nonlinear advection term $\nabla|\vec{v}|^2$ will brings us considerable obstacles when deriving the uniform-in-ε estimates for $(u^\varepsilon, \vec{v}^\varepsilon)$.

We conclude this section by briefly recalling other abundant results obtained for the transformed KS system (1.3) from various angles and hence for the original KS system (1.1) via transformation (1.2). In one dimension, the large time behavior of solutions was investigated when $\Omega = \mathbb{R}$ in [19, 35] with $\varepsilon = 0$ and in [47, 54] with $\varepsilon > 0$. When $\Omega = (0,1)$,the global existence and asymptotics of solutions under Neumann-Dirichlet boundary conditions for $\varepsilon = 0$ were obtained in [39, 72], and later was extended to the case $\varepsilon > 0$ in [62, 67]. For the Dirichlet boundary conditions, the global dynamics of solutions was exploited in [37]. Furthermore the existence and stability of traveling wave solutions were studied in [2, 27, 38, 40–43]. To the best of our knowledge, the known well-posedness results in multi-dimension are merely confined to local large and global small solutions, see [7, 20, 34, 53, 66] for $\Omega = \mathbb{R}^d$ $(d \geq 2)$ and [39] for $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ bounded. Recently the well-posedness of the transformed KS system (1.3) with fractional diffusion has been studied in [16, 17] for $\varepsilon = 0$ where the gradient term ∇u was replaced by a more general term $\nabla u^r (1 \leq r \leq 2)$ in the second equation of (1.3).

The rest of this paper is organized as follows. In section 2, we first present the outer and inner layer profiles and then state our main results on the stability of boundary layer solutions of the transformed system (1.3) as well as the original KS system (1.1). In section 3, we present and prove some necessary regularity results on the outer and inner layer profiles required to prove our main results. Then in section 4, we reformulate our problem and prove the main results. Finally in section 5 (Appendix), we outline the proofs for the outer/inner layer profiles announced in section 2.

2. Notation and main results

Notations.

- Without loss of generality, we assume $0 \le \varepsilon < 1$ since we are concerned with the diffusion limit as $\varepsilon \to 0$. We denote by C and C_0 generic constants that may change from one line to another with C independent of ε but depending on T, and C_0 independent of both ε and T.
- $\mathbb{N}_+$ represents the set of positive integers and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$. For $z \in (0,\infty)$, we denote $\langle z\rangle = \sqrt{z^2+1}$.
- With $1 \le p \le \infty$, we use L^p_{xy} and L^p_{xz} to denote the Lebesgue space $L^p(\mathbb{R}\times\mathbb{R}_+)$ with respect to (x,y) and (x,z), respectively, with corresponding norms $\|\cdot\|_{L^p_{xy}}$ and $\|\cdot\|_{L^p_{xz}}$.
- Similarly, H^k_{xy} and H^k_{xz} for $k \in \mathbb{N}$ represent the Sobolev space $W^{k,2}(\mathbb{R}\times\mathbb{R}_+)$ with respect to (x,y) and (x,z) respectively, with corresponding norms $\|\cdot\|_{H^k_{xy}}$ and $\|\cdot\|_{H^k_{xz}}$. Without confusion, we still use H^k_{xy} and L^p_{xy} to denote the two-dimensional vector spaces $(H^k_{xy})^2$ and $(L^p_{xy})^2$.
- For $k,m \in \mathbb{N}$, we introduce the anisotropic Sobolev space

$$H^k_xH^m_z := \left\{ f(x,z) \in L^2(\mathbb{R}\times\mathbb{R}_+) \,|\, \sum_{0\le l_1\le k,\, 0\le l_2\le m} \|\partial_x^{l_1}\partial_z^{l_2} f(x,z)\|_{L^2_{xz}} < \infty \right\}$$

  with norm $\|\cdot\|_{H^k_xH^m_z}$. Similarly $H^k_xH^m_y$ will be used if the dependent variable of f is $(x,y) \in \mathbb{R}\times\mathbb{R}_+$.
- For simplicity, we use $\|\cdot\|_{L^q_TX}$ $(1 \le q \le \infty)$ to denote $\|\cdot\|_{L^q([0,T];X)}$ for Banach space X.

2.1. Equations for inner and outer layer profiles.

This subsection is devoted to deriving the equations for outer and inner layer profiles by applying formal asymptotic analysis to solutions $(u^\varepsilon, \vec{v}^\varepsilon)$ of (1.3) with (1.7) with small $\varepsilon > 0$. Hence based on the WKB theory (see e.g. [24], [22, Chapter 4], [18, 58]), the solution $(u^\varepsilon, \vec{v}^\varepsilon)$ has the following asymptotic expansions with respect to ε in Ω for $j \in \mathbb{N}$:

$$\begin{aligned} u^\varepsilon(x,y,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2}\left(u^{I,j}(x,y,t) + u^{B,j}(x,z,t)\right), \\ \vec{v}^\varepsilon(x,y,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2}\left(\vec{v}^{I,j}(x,y,t) + \vec{v}^{B,j}(x,z,t)\right), \end{aligned} \tag{2.1}$$

where the boundary layer coordinate is defined as:

$$z = \frac{y}{\varepsilon^{1/2}}, \quad y \in (0,\infty). \tag{2.2}$$

Each term in (2.1) is assumed to be smooth and the boundary-layer profiles $(u^{B,j}, \vec{v}^{B,j})$ enjoy the following basic hypothesis (see also [22, Chapter 4], [18], [58]):

(H) $u^{B,j}$ and $\vec{v}^{B,j}$ decay to zero exponentially as $z \to \infty$.

In order to obtain the equations for outer and inner layer profiles in (2.1), the analysis will be split into three steps. First the initial and boundary values follow from the substitution of (2.1) into the third equality of (1.3) and (1.7). Then we deduce the equations for layer profiles by inserting (2.1) into the first and second equations of (1.3) successively. Applying these procedures and using the asymptotic matching method (details are given in appendix) we deduce that the leading-order outer layer profile $(u^{I,0},\vec{v}^{I,0})(x,y,t)$ satisfies the following initial-boundary value problem:

$$\begin{cases} u_t^{I,0}-\nabla\cdot(u^{I,0}\vec{v}^{\,I,0})=\Delta u^{I,0}, & (x,y,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T),\\ \vec{v}_t^{\,I,0}-\nabla u^{I,0}=0, \\ (u^{I,0},\vec{v}^{\,I,0})(x,y,0)=(u_0,\vec{v}_0)(x,y), \\ u^{I,0}(x,0,t)=\bar{u}(x,t). \end{cases} \tag{2.3}$$

Note that (2.3) is exactly the system (1.3), (1.7) with $\varepsilon=0$, whose solution is denoted as $(u^0,\vec{v}^0)(x,y,t)$. Then we conclude that

$$(u^{I,0},\vec{v}^{\,I,0})(x,y,t)=(u^0,\vec{v}^0)(x,y,t),\quad (x,y,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T) \tag{2.4}$$

thanks to the uniqueness of solutions. The leading-order inner layer profile $u^{B,0}(x,z,t)$ satisfies

$$u^{B,0}(x,z,t)\equiv 0$$

and $v_1^{B,0}(x,z,t)$, the first component of $\vec{v}^{B,0}(x,z,t)$, solves

$$\begin{cases} \partial_t v_1^{B,0}=\partial_z^2 v_1^{B,0}, & (x,z,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T),\\ v_1^{B,0}(x,z,0)=0, \\ \partial_z v_1^{B,0}(x,0,t)=0, \end{cases} \tag{2.5}$$

which gives rise to

$$v_1^{B,0}(x,z,t)\equiv 0, \tag{2.6}$$

by the uniqueness of solutions. The second component of $\vec{v}^{B,0}(x,z,t)$ fulfills

$$\begin{cases} \partial_t v_2^{B,0}+\bar{u}(x,t)v_2^{B,0}=\partial_z^2 v_2^{B,0}, & (x,z,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T),\\ v_2^{B,0}(x,z,0)=0, \\ v_2^{B,0}(x,0,t)=\bar{v}(x,t)-v_2^{I,0}(x,0,t) \end{cases} \tag{2.7}$$

and the first-order inner layer profile $u^{B,1}(x,z,t)$ is determined by $v_2^{B,0}(x,z,t)$ via

$$u^{B,1}(x,z,t)=\bar{u}(x,t)\int_z^\infty v_2^{B,0}(x,\eta,t)\,d\eta. \tag{2.8}$$

Moreover, the first-order outer layer profile $(u^{I,1},\vec{v}^{I,1})(x,y,t)$ is the solution of

$$\begin{cases} u_t^{I,1}=\nabla\cdot(u^{I,0}\vec{v}^{I,1})+\nabla\cdot(u^{I,1}\vec{v}^{\,I,0})+\Delta u^{I,1}, & (x,y,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T),\\ \vec{v}_t^{I,1}=\nabla u^{I,1}, \\ (u^{I,1},\vec{v}^{I,1})(x,y,0)=(0,0), \\ u^{I,1}(x,0,t)=-\bar{u}(x,t)\displaystyle\int_0^\infty v_2^{B,0}(x,z,t)\,dz. \end{cases} \tag{2.9}$$

For the first-order inner layer profile $\vec{v}^{B,1}(x,z,t)$, its first component $v_1^{B,1}(x,z,t)$ satisfies

$$\begin{cases} \partial_t v_1^{B,1}-\partial_x u^{B,1}=\partial_z^2 v_1^{B,1}, & (x,z,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T),\\ v_1^{B,1}(x,z,0)=0, \\ \partial_z v_1^{B,1}(x,0,t)=\partial_x\bar{v}(x,t)-\partial_y v_1^{I,0}(x,0,t) \end{cases} \tag{2.10}$$

and its second component $v_2^{B,1}(x,z,t)$ solves

$$\begin{cases} \partial_t v_2^{B,1} + \bar{u}(x,t) v_2^{B,1} = \partial_z^2 v_2^{B,1} - 2(v_2^{I,0}(x,0,t) + v_2^{B,0})\partial_z v_2^{B,0} + \int_z^\infty \Gamma(x,\eta,t)\,d\eta, \\ v_2^{B,1}(x,z,0) = 0, \qquad (x,z) \in \mathbb{R}\times\mathbb{R}_+, \\ v_2^{B,1}(x,0,t) = -v_2^{I,1}(x,0,t). \end{cases} \tag{2.11}$$

The second-order inner layer profile $u^{B,2}(x,z,t)$ is given as

$$u^{B,2}(x,z,t) = \bar{u}(x,t)\int_z^\infty v_2^{B,1}(x,\eta,t)\,d\eta - \int_z^\infty \int_\eta^\infty \Gamma(x,\xi,t)\,d\xi\,d\eta, \tag{2.12}$$

where

$$\begin{aligned} \Gamma(x,z,t) :=& (u^{I,1}(x,0,t) + u^{B,1})\partial_z v_2^{B,0} + \partial_y u^{I,0}(x,0,t) v_2^{B,0} \\ &+ \partial_z u^{B,1}(v_2^{I,0}(x,0,t) + v_2^{B,0}) + z\partial_y u^{I,0}(x,0,t)\partial_z v_2^{B,0}. \end{aligned} \tag{2.13}$$

Finally $v_1^{B,2}(x,z,t)$, the first component of $\vec{v}^{B,2}(x,z,t)$, solves the following problem:

$$\begin{cases} \partial_t v_1^{B,2} = -\partial_x[2v_2^{I,0}(x,0,t)v_2^{B,0} + v_2^{B,0}v_2^{B,0}] + \partial_x u^{B,2} + \partial_z^2 v_1^{B,2}, \\ v_1^{B,2}(x,z,0) = 0, \qquad (x,z)\in\mathbb{R}\times\mathbb{R}_+, \\ \partial_z v_1^{B,2}(x,0,t) = -\partial_y v_1^{I,1}(x,0,t). \end{cases} \tag{2.14}$$

The derivation of (2.3)-(2.14) will be detailed in Appendix and their well-posedness will be gradually discussed in section 3. One can go further to deduce the initial boundary value problems for higher order layer profiles $(u^{I,j}, v^{I,j})$, $(u^{B,j+1}, v_1^{B,j+1}, v_2^{B,j})$ with $j \geq 2$, but they are not needed to conclude our results.

2.2. **Main results.** It is well-known that the appropriate compatibility conditions for initial and boundary data are necessary to obtain the boundary layer solution and prove its stability (cf. [12, 24, 60]). Following the convention of [31], by "the compatibility conditions up to order m ($m\in\mathbb{N}$) for problem (1.3), (1.7) with $\varepsilon = 0$", we mean that $\partial_t^k u|_{t=0} = \partial_t^k \bar{u}(x,0)$ on the boundary $\partial\Omega = \{(x,y)\in\mathbb{R}^2 |\, y = 0\}$ for $0\le k\le m$, where $\partial_t^k u|_{t=0}$ are determined by u_0, $\vec{v}_0$, $\bar{u}$, $\bar{v}$ and their time derivatives through the equations in (1.3). Specifically in our present work we shall need the following compatibility conditions:

$$(A1)\begin{cases} \bar{u}(x,0) =& u_0(x,0), \\ \partial_t\bar{u}(x,0) =& [\nabla\cdot(u_0\vec{v}_0) + \Delta u_0](x,0), \\ \partial_t^2\bar{u}(x,0) =& \nabla\cdot[\partial_t\bar{u}(x,0)\vec{v}_0(x,0)] + \nabla\cdot[u_0\nabla u_0] + \Delta\partial_t\bar{u}(x,0), \\ \partial_t^3\bar{u}(x,0) =& \nabla\cdot[\partial_t^2\bar{u}(x,0)\vec{v}_0(x,0)] + 2\nabla\cdot[\partial_t\bar{u}(x,0)\nabla u_0] + \nabla\cdot[u_0\nabla\partial_t\bar{u}(x,0)] + \Delta\partial_t^2\bar{u}(x,0), \\ \partial_t^4\bar{u}(x,0) =& \nabla\cdot[\partial_t^3\bar{u}(x,0)\vec{v}_0(x,0)] + 3\nabla\cdot[\partial_t^2\bar{u}(x,0)\nabla u_0(x,0)] \\ & + 3\nabla\cdot[\partial_t\bar{u}(x,0)\nabla\partial_t\bar{u}(x,0)] + \nabla\cdot[u_0\nabla\partial_t^2\bar{u}(x,0)\vec{v}_0] \end{cases}$$

and

$$(A2)\begin{cases} \bar{v}(x,0) =& v_{02}(x,0), \\ \partial_t\bar{v}(x,0) =& \partial_y u_0(x,0), \\ \partial_t^2\bar{v}(x,0) =& \partial_y[\nabla\cdot(u_0\vec{v}_0) + \Delta u_0](x,0), \\ \partial_t^3\bar{v}(x,0) =& \partial_y[\nabla\cdot(\nabla\cdot(u_0\vec{v}_0) + \Delta u_0)\vec{v}_0](x,0) + \nabla\cdot(u_0\vec{v}_0)(x,0) + \Delta[\nabla\cdot(u_0\vec{v}_0) + \Delta u_0](x,0), \end{cases}$$

where (A1) stands for the compatibility condition for problem (1.3), (1.7) with $\varepsilon = 0$ up to order 4 and (A2) for problem (2.7) up to order 3. They can be derived from (2.3) and (2.7). Similarly the compatibility conditions for other initial-boundary problems mentioned in the sequel are defined in the same way (cf. [31, page 319]).

To prove our result, we need the following regularity on solutions of (1.3), (1.7) with $\varepsilon = 0$.

Proposition 2.1. *Assume that the initial and boundary data satisfy*

$$u_0, \vec{v}_0 \in H^9_{xy},\ u_0 \geq 0, \nabla \times \vec{v}_0 = 0; \qquad \partial_t^k \bar{u},\ \partial_t^k \bar{v} \in L^2_{loc}([0,\infty); H_x^{10-2k}) \quad \text{with } 0 \leq k \leq 5$$

and $(A1)$ *hold. Then there exists a time* T_0 *with* $0 < T_0 < \infty$ *such that the problem* (1.3)*,* (1.7) *with* $\varepsilon = 0$ *has a unique solution* $(u^0, \vec{v}^0)(x,y,t)$ *on* $[0,T_0]$ *satisfying* $\nabla \times \vec{v}^0(x,y,t) \equiv 0$ *and*

$$\begin{aligned} &\partial_t^k u^0 \in L^2([0,T_0]; H^{10-2k}_{xy}), \quad k = 0,1,2,3,4,5;\\ &\partial_t^k \vec{v}^0 \in L^2([0,T_0]; H^{11-2k}_{xy}), \quad k = 1,2,3,4,5;\\ &\vec{v}^0 \in L^\infty([0,T_0]; H^9_{xy}). \end{aligned}$$

The proof of Proposition 2.1 is standard and hence omitted for brevity. The interested reader may be referred to [36, Theorem 1.1] where the local well-posedness of (1.3) with $\Omega = \mathbb{R}^d$ $(d \geq 2)$ is proved.

Remark 2.1. Proposition 2.1 only gives the local existence of large solutions to the problem (1.3), (1.7) with $\varepsilon = 0$. In the sequel, we shall denote the maximal lifespan of solutions to (1.3), (1.7) with $\varepsilon = 0$ by $T_{\max} (0 < T_{\max} < \infty)$ without further clarification. The global existence of large solutions to the problem (1.3), (1.7) with $\varepsilon \geq 0$ still remains open to date. However if some smallness conditions are imposed on the initial data $(u_0, \vec{v}_0)$, the global existence of solutions can be obtained (cf. [57]). Furthermore the regularity of initial data can be reduced if we only seek the existence of solutions without exploring convergence of boundary layers which requires higher regularity on solutions.

We are now in a position to state our main result. For brevity, instead of proving (1.8), we shall prove a similar result with convergence rate for $\vec{v}$ by $\mathscr{O}(\varepsilon^{1/4})$, and remark that (1.8) can be obtained similarly by imposing a higher regularity on the initial and boundary data.

Theorem 2.1. *Suppose that the initial and boundary data satisfy*

$$u_0, \vec{v}_0 \in H^9_{xy},\ u_0 \geq 0, \nabla \times \vec{v}_0 = 0; \qquad \partial_t^k \bar{u},\ \partial_t^k \bar{v} \in L^2_{loc}([0,\infty); H_x^{10-2k}) \quad \text{with } 0 \leq k \leq 5$$

and the compatibility condition $(A1)-(A2)$*. Let* $(u^0, \vec{v}^0)(x,y,t)$ *be the solution derived in Proposition 2.1 and let* $0 < T \leq T_{\max}$*. Then there exists a constant* $\varepsilon_T > 0$ *decreasing in* T *with* $\lim_{T\to\infty} \varepsilon_T = 0$ *(defined in Lemma 4.3) such that for any* $\varepsilon \in (0, \varepsilon_T]$*, the problem* (1.3)*,* (1.7) *admits a unique solution* $(u^\varepsilon, \vec{v}^\varepsilon) \in C([0,T]; H^2_{xy} \times H^2_{xy})$ *on* $[0,T]$ *satisfying* $\nabla \times \vec{v}^\varepsilon(x,y,t) \equiv 0$ *and*

$$\begin{aligned} &\|u^\varepsilon(x,y,t) - u^0(x,y,t)\|_{L^\infty([0,T];L^\infty_{xy})} \leq C\varepsilon^{1/2},\\ &\|\vec{v}^\varepsilon(x,y,t) - \vec{v}^0(x,y,t) - \left(0, v_2^{B,0}\right)\left(x, \frac{y}{\sqrt{\varepsilon}}, t\right)\|_{L^\infty([0,T];L^\infty_{xy})} \leq C\varepsilon^{1/4}, \end{aligned} \tag{2.15}$$

where the constant $C > 0$ *is independent of* ε *and*

$$v_2^{B,0}(x,z,t) := \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-\eta)^2}{4(t-s)} + (t-s)\bar{u}\right)} [\bar{u}(\bar{v} - v_2^0(x,0,s) - \partial_s v_2^0(x,0,s))] d\eta ds. \tag{2.16}$$

Remark 2.2. The convergence rate for $\vec{v}$ in (2.15) can be enhanced to $\mathscr{O}(\varepsilon^{1/2})$ by first including the higher-order profiles $(u^{I,2}, \vec{v}^{I,2})$, $(u^{B,3}, v_1^{B,3}, v_2^{B,2})$ in the approximation $(U^a, \vec{V}^a)$ (see Section 4), and then applying the similar procedures as proving (2.15) based on a stronger assumption on initial-boundary data: $u^0, \vec{v}^0 \in H^{11}$, $\partial_t^k \bar{u}$, $\partial_t^k \bar{v} \in L^2_{loc}([0,\infty]; H_x^{12-2k})$.

Remark 2.3. The regularity of $(u^\varepsilon, \vec{v}^\varepsilon)$ in Theorem 2.1 is much lower than that of the given initial data $(u_0, \vec{v}_0) \in H^9_{xy}$, since the conditions $(A1)$-$(A2)$ only provide the zero-th order compatibility condition for problem (1.3), (1.7) with $\varepsilon > 0$ (i.e. $\bar{u}(x,0) = u_0(x,0)$ and $\bar{v}(x,0) = v_{02}(x,0)$). By assuming further that the initial-boundary data satisfy the compatibility conditions of (1.3), (1.7) (with $\varepsilon > 0$) up to order 4, the regularity space of $(u^\varepsilon, \vec{v}^\varepsilon)$ can be improved to $C([0,T]; H^9_{xy} \times H^9_{xy})$. However the regularity derived in Theorem 2.1 is sufficient to derive our main result (2.15).

Finally we transfer the results obtained in Theorem 2.1 to the original KS chemotaxis system (1.1). Note that the boundary condition in (1.7) for $\vec{v}$ is equivalent to $[\nabla c \cdot \vec{n} + \bar{v}(x,t)c]|_{y=0} = 0$ by a simple calculation, where $\vec{n}$ denotes the unit outward normal vector of $\partial\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, y = 0\}$, namely $\vec{n} = (0,-1)$. Then the corresponding initial-boundary value problem of the original chemotaxis model (1.1) reads as

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi \frac{u}{c} \nabla c), & (x,y,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T), \\ c_t = \varepsilon \Delta c - uc, \\ (u,c)(x,y,0) = (u_0, c_0)(x,y), \\ u|_{y=0} = \bar{u}(x,t), \quad [\nabla c \cdot \vec{n} + \bar{v}(x,t)c]|_{y=0} = 0 & \text{if } \varepsilon > 0, \\ u|_{y=0} = \bar{u}(x,t) & \text{if } \varepsilon = 0. \end{cases} \tag{2.17}$$

By Theorem 2.1, we get the following results for the problem (2.17).

Theorem 2.2. *Suppose $(u_0, \ln c_0) \in H^9_{xy} \times H^{10}_{xy}$ with $u_0 \geq 0$, $c_0 > 0$. Let the assumptions in Theorem 2.1 hold with $\vec{v}_0 = -\frac{\nabla c_0}{c_0}$. Then (2.17) admits a unique solution $(u^\varepsilon, c^\varepsilon) \in C([0,T]; H^2_{xy} \times H^3_{xy})$ for $\varepsilon \in (0, \varepsilon_T]$ and $(u^0, c^0) \in C([0,T]; H^9_{xy} \times H^{10}_{xy})$ for $\varepsilon = 0$ such that*

$$\begin{aligned} &\|u^\varepsilon(x,y,t) - u^0(x,y,t)\|_{L^\infty([0,T];L^\infty_{xy})} \leq C\varepsilon^{1/2}, \\ &\|c^\varepsilon(x,y,t) - c^0(x,y,t)\|_{L^\infty([0,T];L^\infty_{xy})} \leq C\varepsilon^{1/4} \end{aligned} \tag{2.18}$$

and

$$\|\nabla c^\varepsilon(x,y,t) - \nabla c^0(x,y,t) + \big(0,\ c^0(x,y,t) v_2^{B,0}\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big)\big)\|_{L^\infty([0,T];L^\infty_{xy})} \leq C\varepsilon^{1/4}, \tag{2.19}$$

where $v_2^{B,0}$ is defined in (2.16) and the constant $C > 0$ is independent of ε.

The results of Theorem 2.2 show that the boundary layers will be present in the slope (derivative) of solution component c (i.e. ∇c) instead of the value of c itself. The first equation of (2.17) indicates that the presence of boundary layer in ∇c will cause a rapid change in chemotactic flux near the boundary for small $\varepsilon > 0$. This means that chemical diffusion rate ε plays an important role for the dynamics in the vicinity of boundary and can not be neglected, which elucidates the question whether the dynamics of (1.1) is significantly different between $\varepsilon = 0$ and $\varepsilon > 0$ small.

3. Regularity of outer and inner layer profiles

To assert the well-posedness of solutions of (2.7)-(2.14), we first exploit some preliminary results. In particular, to solve (2.7) and (2.11) we introduce the following auxiliary system

$$\begin{cases} \theta_t(x,z,t) + \bar{u}(x,t)\theta(x,z,t) = \partial_z^2 \theta(x,z,t) + \rho(x,z,t), & (x,z,t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \theta(x,z,0) = 0, \\ \theta|_{z=0} = 0. \end{cases} \tag{3.1}$$

Then the following regularity result on solutions of (3.1) holds.

Proposition 3.1. *Let $0 < T < \infty$ and $m \in \mathbb{N}_+$. Suppose ρ satisfies for all $l \in \mathbb{N}$ that*

$$\langle z\rangle^l \partial_t^k \rho \in L^2([0,T]; H_x^{2m-2k} L_z^2), \quad k = 0,1,\cdots,m$$

and $\bar{u}(x,t)$ satisfies

$$\partial_t^k \bar{u} \in L^2([0,T]; H_x^{2m+1-2k}), \quad k = 0,1,\cdots,m.$$

Assume further that ρ and $\bar{u}$ satisfy the compatibility conditions up to order $(m-1)$ for the problem (3.1). *Then* (3.1) *admits a unique solution $\theta(x,z,t)$ on $[0,T]$ such that for any $l \in \mathbb{N}$*

$$\langle z\rangle^l \partial_t^k \theta \in L^\infty([0,T]; H_x^{2m-2k} H_z^1) \cap L^2([0,T]; H_x^{2m-2k} H_z^2), \quad k = 0,1,\cdots,m.$$

We omit the proof of Proposition 3.1 since it is standard and refer the reader to [9, page 380-388] for details. To study (2.9) we consider the following initial-boundary problem

$$\begin{cases} h_t = \Delta h + \nabla\cdot(\vec{f}_1 h) + \nabla\cdot(f_2 \vec{w}) + f, & (x,y,t) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+, \\ \vec{w}_t = \nabla h + \vec{g}, \\ (h,\vec{w})(x,y,0) = (h_0,\vec{w}_0)(x,y), \\ h|_{y=0} = 0, \end{cases} \tag{3.2}$$

whose well-posedness is as follows.

Proposition 3.2. *Let $0 < T < \infty$ and $m \in \mathbb{N}_+$. Suppose that $(h_0,\vec{w}_0) \in H_{xy}^{2m-1} \times H_{xy}^{2m-1}$ and*

$$\partial_t^k f \in L^2([0,T]; H_{xy}^{2m-2-2k}), \quad \partial_t^k \vec{g} \in L^2([0,T]; H_{xy}^{2m-1-2k}) \quad \text{for } k = 0,1,\cdots,m-1;$$
$$\partial_t^k \vec{f}_1 \in L^\infty([0,T]; H_{xy}^{2m-1-2k}), \quad \partial_t^k f_2 \in L^2([0,T]; H_{xy}^{2m-2k}) \quad \text{for } k = 0,1,\cdots,m-1.$$

Assume further that $(h_0,\vec{w}_0)$ and f, $\vec{g}$, $\vec{f}_1$, f_2 satisfy the compatibility conditions up to order $(m-1)$ for problem (3.2). *Then* (3.2) *admits a unique solution $(h,\vec{w})(x,y,t)$ on $[0,T]$ such that*

$$\partial_t^k h \in L^2([0,T]; H_{xy}^{2m-2k}) \quad \text{for } k = 0,1,\cdots,m;$$
$$\partial_t^k \vec{w} \in L^2([0,T]; H_{xy}^{2m+1-2k}) \quad \text{for } k = 1,\cdots,m; \qquad \vec{w} \in L^\infty([0,T]; H_{xy}^{2m-1}).$$

The proof of Proposition 3.2 is omitted for brevity and refer to [24, Proposition 3.1] for details.

Finally, for the regularity on solutions of (2.10) and (2.14), we introduce the following system

$$\begin{cases} \psi_t(x,z,t) = \partial_z^2 \psi(x,z,t) + r(x,z,t), & (x,z,t) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+, \\ \psi(x,z,0) = 0, \\ \partial_z \psi(x,0,t) = s(x,t). \end{cases} \tag{3.3}$$

For system (3.3), we have the following result.

Proposition 3.3. *Let $0 < T < \infty$ and assume the integer $m \geq 3$. Assume $r(x,z,t)$ fulfills for all $l \in \mathbb{N}$ that*

$$\langle z\rangle^l r,\ \langle z\rangle^l \partial_t r \in L^2([0,T]; H_x^m L_z^2); \qquad \langle z\rangle^l \partial_t^2 r \in L^2([0,T]; H_x^{m-2} L_z^2)$$

and $s(x,t)$ satisfies

$$s,\ \partial_t s \in L^2([0,T]; H_x^m); \qquad \partial_t^2 s \in L^2([0,T]; H_x^{m-2}).$$

Assume further that r and s satisfy the compatibility conditions up to order 1 for the initial-boundary problem (3.3). *Then there exists a unique solution $\psi(x,z,t)$ of* (3.3) *on $[0,T]$ such* *that for any $l \in \mathbb{N}$:*

$$\begin{aligned}&\langle z\rangle^l \psi,\ \langle z\rangle^l \partial_z \psi,\ \langle z\rangle^l \partial_t \psi \in L^\infty([0,T];H_x^m L_z^2)\cap L^2([0,T];H_x^m H_z^1);\\ &\langle z\rangle^l \partial_z\partial_t \psi,\ \langle z\rangle^l \partial_t^2 \psi \in L^\infty([0,T];H_x^{m-2} L_z^2)\cap L^2([0,T];H_x^{m-2} H_z^1).\end{aligned} \tag{3.4}$$

Proof. The existence and uniqueness for solution of system (3.3) directly follows from [31, page 170, Theorem 5.1] and we omit it for brevity. It remains to get the desired regularity estimates (3.4) for solution ψ. With $0 \le j \le m$ and $l \in \mathbb{N}$, we first apply ∂_x^j (j-th order differentiation) to (3.3), then multiply the resulting equation with $2\langle z\rangle^{2l}\partial_x^j \psi$ in L^2_{xz} and use integration by parts to derive

$$\begin{aligned}&\frac{d}{dt}\|\langle z\rangle^l \partial_x^j \psi\|_{L^2_{xz}}^2 + 2\|\langle z\rangle^l \partial_x^j \partial_z \psi\|_{L^2_{xz}}^2\\ &= -4l\int_0^\infty\int_{-\infty}^\infty \langle z\rangle^{2l-2} z(\partial_z\partial_x^j\psi)(\partial_x^j\psi)dxdz + 2\int_0^\infty\int_{-\infty}^\infty \langle z\rangle^{2l}(\partial_x^j r)(\partial_x^j \psi)dxdz\\ &\quad + 2\int_{-\infty}^\infty (\partial_x^j\partial_z\psi(x,0,t))(\partial_x^j\psi(x,0,t))dx\\ &\le \frac{1}{2}\|\langle z\rangle^l \partial_x^j \partial_z\psi\|_{L^2_{xz}}^2 + C_0(l^2+1)\|\langle z\rangle^l \partial_x^j\psi\|_{L^2_{xz}}^2 + \|\langle z\rangle^l \partial_x^j r\|_{L^2_{xz}}^2\\ &\quad + 2\int_{-\infty}^\infty (\partial_x^j s(x,t))(\partial_x^j\psi(x,0,t))dx\end{aligned} \tag{3.5}$$

with

$$\begin{aligned}2\int_{-\infty}^\infty (\partial_x^j s(x,t))(\partial_x^j\psi(x,0,t))dx &\le 2\int_{-\infty}^\infty |\partial_x^j s(x,t)|\,\|\partial_x^j\psi(x,z,t)\|_{L_z^\infty}dx\\ &\le C_0\int_{-\infty}^\infty |\partial_x^j s(x,t)|\,\|\partial_x^j\psi(x,z,t)\|_{H_z^1}dx\\ &\le \frac{1}{2}\|\langle z\rangle^l\partial_x^j\partial_z\psi\|_{L^2_{xz}}^2 + \frac{1}{2}\|\langle z\rangle^l\partial_x^j\psi\|_{L^2_{xz}}^2 + C_0\|\partial_x^j s\|_{L_x^2}^2,\end{aligned}$$

where the Sobolev embedding inequality has been used. Summing (3.5) from $j=0$ to $j=m$ and applying Gronwall's inequality, one deduces that

$$\|\langle z\rangle^l\psi\|_{L_T^\infty H_x^m L_z^2}^2 + \|\langle z\rangle^l\partial_z\psi\|_{L_T^2 H_x^m L_z^2}^2 \le C. \tag{3.6}$$

As stated in notations, the C_0 and C are generic constants that may change from one line to another throughout this paper. We proceed to derive higher regularity estimates for ψ. Similar to the above procedure in deriving (3.5), we apply ∂_x^j to (3.3) and multiply the resulting equation with $2\langle z\rangle^{2l}\partial_x^j\partial_t\psi$ in L^2_{xz} to have

$$\begin{aligned}&\frac{d}{dt}\|\langle z\rangle^l \partial_x^j \partial_z\psi\|_{L^2_{xz}}^2 + 2\|\langle z\rangle^l \partial_x^j \partial_t \psi\|_{L^2_{xz}}^2\\ &\le \frac{1}{2}\|\langle z\rangle^l \partial_x^j \partial_t\psi\|_{L^2_{xz}}^2 + C_0(l^2+1)\|\langle z\rangle^l \partial_x^j\partial_z\psi\|_{L^2_{xz}}^2 + C_0\|\langle z\rangle^l \partial_x^j r\|_{L^2_{xz}}^2\\ &\quad + 2\int_{-\infty}^\infty (\partial_x^j s(x,t))(\partial_x^j\partial_t\psi(x,0,t))dx\end{aligned} \tag{3.7}$$

with

$$2\int_{-\infty}^\infty (\partial_x^j s(x,t))(\partial_x^j\partial_t\psi(x,0,t))dx \le \frac{1}{2}\|\langle z\rangle^l\partial_x^j\partial_z\partial_t\psi\|_{L^2_{xz}}^2 + \frac{1}{2}\|\langle z\rangle^l\partial_x^j\partial_t\psi\|_{L^2_{xz}}^2 + C_0\|\partial_x^j s\|_{L_x^2}^2.$$

On the other hand, by setting $t=0$ in the first equation of (3.3) and noting that $\partial_z^2\psi(x,z,0)=0$ thanks to the initial condition $\psi(x,z,0)=0$ in (3.3) we derive $\partial_t\psi(x,z,0)=r(x,z,0)$. Then applying ∂_t to (3.3) one finds that $\partial_t\psi$ solves a similar system as (3.3) with $r(x,z,t)$, $s(x,t)$ and the initial condition replaced by $\partial_t r(x,z,t)$, $\partial_t s(x,t)$ and $\partial_t \psi(x,z,0) = r(x,z,0)$, respectively. Thus it follows from (3.5) that

$$\begin{aligned}&\frac{d}{dt}\|\langle z\rangle^l \partial_x^j \partial_t \psi\|_{L^2_{xz}}^2 + 2\|\langle z\rangle^l \partial_x^j \partial_z \partial_t \psi\|_{L^2_{xz}}^2 \\ \le &\|\langle z\rangle^l \partial_x^j \partial_z \partial_t \psi\|_{L^2_{xz}}^2 + C_0(l^2+1)\|\langle z\rangle^l \partial_x^j \partial_t \psi\|_{L^2_{xz}}^2 + \|\langle z\rangle^l \partial_x^j \partial_t r\|_{L^2_{xz}}^2 + C_0\|\partial_x^j \partial_t s\|_{L^2_x}^2.\end{aligned} \tag{3.8}$$

We add (3.8) to (3.7) and then sum the results from $j=0$ to $j=m$ to get

$$\begin{aligned}&\frac{d}{dt}(\|\langle z\rangle^l \partial_z \psi\|_{H_x^m L_z^2}^2 + \|\langle z\rangle^l \partial_t \psi\|_{H_x^m L_z^2}^2) + \|\langle z\rangle^l \partial_t \psi\|_{H_x^m L_z^2} + \|\langle z\rangle^l \partial_z \partial_t \psi\|_{H_x^m L_z^2}^2 \\ \le &C_0(\|\langle z\rangle^l \partial_z \psi\|_{H_x^m L_z^2}^2 + \|\langle z\rangle^l \partial_t \psi\|_{H_x^m L_z^2}^2) + C_0(\|\langle z\rangle^l r\|_{H_x^m L_z^2}^2 + \|\langle z\rangle^l \partial_t r\|_{H_x^m L_z^2}^2 + \|s\|_{H_x^m}^2 + \|\partial_t s\|_{H_x^m}^2).\end{aligned}$$

which along with Gronwall's inequality leads to

$$\|\langle z\rangle^l \partial_z \psi\|_{L_T^\infty H_x^m L_z^2}^2 + \|\langle z\rangle^l \partial_t \psi\|_{L_T^\infty H_x^m L_z^2}^2 + \|\langle z\rangle^l \partial_t \psi\|_{L_T^2 H_x^m L_z^2} + \|\langle z\rangle^l \partial_z \partial_t \psi\|_{L_T^2 H_x^m L_z^2}^2 \le C. \tag{3.9}$$

By an analogous argument as deriving (3.9) one can deduce for all $l \in \mathbb{N}$ that

$$\begin{aligned}&\|\langle z\rangle^l \partial_z \partial_t \psi\|_{L_T^\infty H_x^{m-2} L_z^2}^2 + \|\langle z\rangle^l \partial_t^2 \psi\|_{L_T^\infty H_x^{m-2} L_z^2}^2 \\ &\quad + \|\langle z\rangle^l \partial_t^2 \psi\|_{L_T^2 H_x^{m-2} L_z^2} + \|\langle z\rangle^l \partial_z \partial_t^2 \psi\|_{L_T^2 H_x^{m-2} L_z^2}^2 \le C.\end{aligned} \tag{3.10}$$

Combining (3.6), (3.9) and (3.10), we get the desired estimates and complete the proof. □

With the above results in hand, we establish the well-posedness of (2.7)-(2.14).

Lemma 3.1. *Suppose the assumptions in Theorem 2.1 hold. Let* $(u^0, \vec{v}^0)(x,y,t)$ *be the solution obtained in Proposition 2.1 and* $0 < T \le T_{\max}$. *Then*

$$v_2^{B,0}(x,z,t) := \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-\eta)^2}{4(t-s)} + (t-s)\bar{u}\right)} [\bar{u}(\bar{v} - v_2^0(x,0,s) - \partial_s v_2^0(x,0,s))] d\eta ds \tag{3.11}$$

is the unique solution of (2.7) *on* $[0,T]$ *satisfying for all* $l \in \mathbb{N}$ *that*

$$\langle z\rangle^l \partial_t^k v_2^{B,0} \in L^\infty([0,T]; H_x^{8-2k} H_z^1) \cap L^2([0,T]; H_x^{8-2k} H_z^2), \quad k = 0,1,2,3,4. \tag{3.12}$$

Furthermore, it follows from the equations (2.7) *and* (2.8) *that*

$$\langle z\rangle^l v_2^{B,0} \in L^\infty([0,T]; H_x^6 H_z^3), \quad \langle z\rangle^l \partial_t v_2^{B,0} \in L^\infty([0,T]; H_x^4 H_z^3) \tag{3.13}$$

and that

$$\langle z\rangle^l \partial_t^k u^{B,1} \in L^\infty([0,T]; H_x^{8-2k} H_z^2) \cap L^2([0,T]; H_x^{8-2k} H_z^3), \quad k = 0,1,2,3,4.$$

Proof. Observing that for fixed $x \in \mathbb{R}$, (2.7) can be converted to the one dimensional heat equation with independent variables $(t,z) \in (0,T) \times \mathbb{R}_+$, which has been explicitly solved by a formula similar to (3.11) using the reflection method with odd extension in [24, Lemma 3.2]. Thus we omit the derivation of (3.11) for brevity and refer the reader to [24, Lemma 3.2] for details. We proceed to prove (3.12). Let $\varphi(z)$ be a smooth function defined on $[0,\infty)$ satisfying

$$\varphi(0) = 1, \qquad \varphi(z) = 0 \ \text{ for } \ z > 1. \tag{3.14}$$

Denote $\tilde{v}_2^{B,0}(x,z,t) = v_2^{B,0}(x,z,t) - \left(\bar{v}(x,t) - v_2^0(x,0,t)\right)\varphi(z)$. Then one deduces from (2.7) and (2.4) that

$$\begin{cases} \partial_t \tilde{v}_2^{B,0} + \bar{u}(x,t)\tilde{v}_2^{B,0} = \partial_z^2 \tilde{v}_2^{B,0} + \rho(x,z,t), \quad (x,z,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T), \\ \tilde{v}_2^{B,0}(x,z,0) = 0, \\ \tilde{v}_2^{B,0}(x,0,t) = 0, \end{cases} \tag{3.15}$$

where $\rho(x,z,t) = \big(\bar{v}(x,t) - v_2^0(x,0,t)\big)\partial_z^2\varphi(z) - \partial_t\big(\bar{v}(x,t) - v_2^0(x,0,t)\big)\varphi(z) - \bar{u}(x,t)\big(\bar{v}(x,t) - v_2^0(x,0,t)\big)\varphi(z)$. The compatibility condition $\bar{v}(x,0) = v_{02}(x,0)$ in $(A2)$ has been used to determine the initial data of $\tilde{v}_2^{B,0}$ in (3.15). We next prove that ρ satisfies the assumptions in Proposition 3.1 with $m=4$. First note that for $f(x,y,t) \in H_{xy}^{k+1}$ with fixed $t>0$ and $k \in \mathbb{N}$ the following holds

$$\begin{aligned}
\|f(x,0,t)\|_{H_x^k}^2 &= \sum_{j=0}^{k}\int_{-\infty}^{\infty}|\partial_x^j f(x,0,t)|^2\,dx \\
&\le \sum_{j=0}^{k}\int_{-\infty}^{\infty}\|\partial_x^j f(x,y,t)\|_{L_y^\infty}^2 dx \\
&\le C_0\sum_{j=0}^{k}\int_{-\infty}^{\infty}\|\partial_x^j f(x,y,t)\|_{H_y^1}^2 dx \le C_0\|f(x,y,t)\|_{H_{xy}^{k+1}}^2,
\end{aligned}\tag{3.16}$$

where the Sobolev embedding inequality has been used. Then it follows from Proposition 2.1 and (3.16) that

$$\|\partial_t^k v_2^0(x,0,t)\|_{L_T^2 H_x^{10-2k}} \le \|\partial_t^k v_2^0\|_{L_T^2 H_{xy}^{11-2k}} \le C, \quad k=1,2,3,4,5 \tag{3.17}$$

and that $\|v_2^0(x,0,t)\|_{L_T^2 H_x^8} \le \|v_2^0\|_{L_T^2 H_{xy}^9} \le C$. Hence from the above estimates we deduce for $l \in \mathbb{N}$ and $k=0,1,2,3,4$ that

$$\begin{aligned}
&\|\langle z\rangle^l \partial_t^k \rho\|_{L_T^2 H_x^{8-2k} L_z^2} \\
\le&\big(\|\partial_t^k \bar{v}\|_{L_T^2 H_x^{8-2k}} + \|\partial_t^k v_2^0(x,0,t)\|_{L_T^2 H_x^{8-2k}}\big)\|\langle z\rangle^l \partial_z^2\varphi\|_{L_z^2} \\
&+\big(\|\partial_t^{k+1}\bar{v}\|_{L_T^2 H_x^{10-2(k+1)}} + \|\partial_t^{k+1} v_2^0(x,0,t)\|_{L_T^2 H_x^{10-2(k+1)}}\big)\|\langle z\rangle^l \varphi\|_{L_z^2} \\
&+\sum_{j=0}^{k}\big(\|\partial_t^j \bar{v}\|_{L_T^2 H_x^{8-2j}} + \|\partial_t^j v_2^0(x,0,t)\|_{L_T^2 H_x^{8-2j}}\big)\|\partial_t^{k-j}\bar{u}\|_{L_T^\infty H_x^{9-2(k-j)}}\|\langle z\rangle^l \varphi\|_{L_z^2} \\
\le& C,
\end{aligned}\tag{3.18}$$

where $\|\partial_t^{k-j}\bar{u}\|_{L_T^\infty H_x^{9-2(k-j)}} \le C$ has been used thanks to the assumptions on $\bar{u}$ in Theorem 2.1. Moreover, it is easy to verify that ρ and $\bar{u}$ satisfy the compatibility conditions up to order 3 for the problem (3.15) under assumption $(A2)$. We then apply Proposition 3.1 with $m=4$ to (3.15) to conclude that

$$\langle z\rangle^l \partial_t^k \tilde{v}_2^{B,0} \in L^\infty([0,T]; H_x^{8-2k} H_z^1) \cap L^2([0,T]; H_x^{8-2k} H_z^2), \quad k=0,1,2,3,4,$$

which along with the definition of $\tilde{v}_2^{B,0}$ and (3.17) gives rise to (3.12). The estimate for $u^{B,1}$ follows directly from (2.8), (3.12) and the assumptions on $\bar{u}$ in Theorem 2.1. It remains to prove (3.13). Indeed, by (2.7) and (3.12) we deduce for all $l \in \mathbb{N}$ that

$$\|\langle z\rangle^l v_2^{B,0}\|_{L_T^\infty H_x^6 H_z^3} \le C_0\big(\|\bar{u}\|_{L_T^\infty H_x^6}\|\langle z\rangle^l v_2^{B,0}\|_{L_T^\infty H_x^6 H_z^1} + \|\langle z\rangle^l \partial_t v_2^{B,0}\|_{L_T^\infty H_x^6 H_z^1}\big) \le C. \tag{3.19}$$

A similar argument gives $\|\langle z\rangle^l \partial_t v_2^{B,0}\|_{L_T^\infty H_x^4 H_z^3} \le C$. The proof is completed. □

Lemma 3.2. *Suppose the assumptions in Theorem 2.1 hold. Let $(u^0,\vec{v}^0)(x,y,t)$ and $v_2^{B,0}(x,z,t)$ be as obtained in Proposition 2.1 and Lemma 3.1, respectively. Then* (2.9) *admits a unique solutions $(u^{I,1},\vec{v}^{I,1})(x,y,t)$ on $[0,T]$ such that*

$$\begin{aligned}
&\partial_t^k u^{I,1} \in L^2([0,T]; H_{xy}^{8-2k}), \quad k=0,1,2,3,4; \\
&\partial_t^k \vec{v}^{I,1} \in L^2([0,T]; H_{xy}^{9-2k}), \quad k=1,2,3,4; \qquad \vec{v}^{I,1} \in L^\infty([0,T]; H_{xy}^7).
\end{aligned}\tag{3.20}$$

Proof. Let φ be as defined in (3.14). We denote $\tilde{u}^{I,1}(x,y,t) = u^{I,1}(x,y,t) + \varphi(y)\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,z,t)dz$. Then it follows from (2.9) that

$$\begin{cases} \partial_t \tilde{u}^{I,1} = \nabla\cdot(\vec{v}^0\tilde{u}^{I,1}) + \nabla\cdot(u^0\vec{v}^{I,1}) + \Delta\tilde{u}^{I,1} + f, \\ \vec{v}_t^{I,1} = \nabla\tilde{u}^{I,1} + \vec{g}, \\ (\tilde{u}^{I,1},\vec{v}^{I,1})(x,y,0) = (0,0), \\ \tilde{u}^{I,1}(x,0,t) = 0, \end{cases} \tag{3.21}$$

where $\vec{g}(x,y,t) = -\nabla\big[\varphi(y)\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,z,t)dz\big]$ and

$$\begin{aligned} f(x,y,t) = &\varphi(y)\partial_t\big[\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,z,t)dz\big] - \Delta\big[\varphi(y)\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,z,t)dz\big] \\ &- \nabla\cdot\big[\varphi(y)\bar{u}(x,t)\vec{v}^0(x,y,t)\int_0^\infty v_2^{B,0}(x,z,t)dz\big]. \end{aligned}$$

To apply Proposition 3.2 with $m=4$ to (3.21) we next verify that $\vec{v}^0$, u^0, f and $\vec{g}$ satisfy the corresponding assumptions. By the Cauchy-Schwarz inequality and Lemma 3.1 we deduce for $j = 0,1,2,3,4$ that

$$\Big\|\int_0^\infty \partial_t^j v_2^{B,0}dz\Big\|_{L_T^\infty H_x^{8-2j}} \le \Big(\int_0^\infty \langle z\rangle^{-2}dz\Big)^{1/2}\|\langle z\rangle\partial_t^j v_2^{B,0}\|_{L_T^\infty H_x^{8-2j}L_z^2} \le C. \tag{3.22}$$

Thus it follows for $k = 0,1,2,3$ that

$$\begin{aligned} \|\partial_t^k f\|_{L_T^2 H_{xy}^{6-2k}} \le &C_0\sum_{j=0}^{k+1}\|\partial_t^{k+1-j}\bar{u}\|_{L_T^2H_x^{8-2(k+1-j)}}\Big\|\int_0^\infty \partial_t^j v_2^{B,0}dz\Big\|_{L_T^\infty H_x^{7-2j}}\|\varphi\|_{H_y^6} \\ &+C_0\sum_{i+j=0}^{k}\|\partial_t^{k-(i+j)}\bar{u}\|_{L_T^2H_x^{7-2(k-i-j)}}\|\partial_t^i\vec{v}^0\|_{L_T^\infty H^{7-2i}}\Big\|\int_0^\infty \partial_t^j v_2^{B,0}dz\Big\|_{L_T^\infty H_x^{7-2j}}\|\varphi\|_{H_y^7} \\ &+C_0\sum_{j=0}^{k}\|\partial_t^{k-j}\bar{u}\|_{L_T^2H_x^{8-2(k-j)}}\Big\|\int_0^\infty \partial_t^j v_2^{B,0}dz\Big\|_{L_T^\infty H_x^{8-2j}}\|\varphi\|_{H_y^8} \le C. \end{aligned}$$

Similarly, for $k = 0,1,2,3$, one gets $\|\partial_t^k\vec{g}\|_{L_T^2H_{xy}^{7-2k}} \le C$.

It is easy to verify that f, $\vec{g}$, u^0 and $\vec{v}^0$ satisfy the compatibility conditions up to order 3 for problem (3.21) under assumption $(A1)$-$(A2)$. By the above estimates for $\vec{g}$, f and Proposition 2.1, we apply Proposition 3.2 with $m=4$ to (3.21) to conclude that

$$\begin{aligned} &\partial_t^k\tilde{u}^{I,1} \in L^2([0,T];H_{xy}^{8-2k}), \quad k = 0,1,2,3,4; \\ &\partial_t^k\vec{v}^{I,1} \in L^2([0,T];H_{xy}^{9-2k}), \quad k = 1,2,3,4 \qquad \text{and} \quad \vec{v}^{I,1} \in L^\infty([0,T];H_{xy}^7), \end{aligned}$$

which, along with the definition of $\tilde{u}^{I,1}$ and (3.22), leads to (3.20) and completes the proof. □

Lemma 3.3. *Suppose the assumptions in Theorem 2.1 hold true. Let $(u^0,\vec{v}^0)(x,y,t)$ and $u^{B,1}(x,z,t)$ be as derived in Proposition 2.1 and Lemma 3.1, respectively. Then there exists a unique solution $v_1^{B,1}(x,z,t)$ of (2.10) on $[0,T]$ such that for any $l \in \mathbb{N}$*

$$\begin{aligned} &\langle z\rangle^l v_1^{B,1},\ \langle z\rangle^l\partial_z v_1^{B,1},\ \langle z\rangle^l\partial_t v_1^{B,1} \in L^\infty([0,T];H_x^5L_z^2)\cap L^2([0,T];H_x^5H_z^1); \\ &\langle z\rangle^l\partial_z\partial_t v_1^{B,1},\ \langle z\rangle^l\partial_t^2 v_1^{B,1} \in L^\infty([0,T];H_x^3L_z^2)\cap L^2([0,T];H_x^3H_z^1). \end{aligned} \tag{3.23}$$

Furthermore, it follows from (2.10) *that*

$$\langle z\rangle^l v_1^{B,1} \in L^\infty([0,T];H_x^5H_z^2), \quad \langle z\rangle^l\partial_t v_1^{B,1} \in L^\infty([0,T];H_x^3H_z^2). \tag{3.24}$$

Proof. Let $r(x,z,t)=\partial_x u^{B,1}(x,z,t)$ and $s(x,t)=\partial_x\bar{v}(x,t)-\partial_y v_1^0(x,0,t)$. We next verify that $r(x,z,t)$ and $s(x,t)$ satisfy the assumptions in Proposition 3.3 with $m=5$. In fact, for $l\in\mathbb{N}$ one deduces from Lemma 3.1 that

$$\begin{aligned}&\|\langle z\rangle^l r\|_{L_T^2H_x^5L_z^2}+\|\langle z\rangle^l\partial_t r\|_{L_T^2H_x^5L_z^2}+\|\langle z\rangle^l\partial_t^2 r\|_{L_T^2H_x^3L_z^2}\\ \le&\|\langle z\rangle^l u^{B,1}\|_{L_T^2H_x^6L_z^2}+\|\langle z\rangle^l\partial_t u^{B,1}\|_{L_T^2H_x^6L_z^2}+\|\langle z\rangle^l\partial_t^2 u^{B,1}\|_{L_T^2H_x^4L_z^2}\le C.\end{aligned}$$

Moreover, (3.16) and Proposition 2.1 entail that

$$\begin{aligned}\|s\|_{L_T^2H_x^5}+\|\partial_t s\|_{L_T^2H_x^5}+\|\partial_t^2 s\|_{L_T^2H_x^3}\le&\|\bar{v}\|_{L_T^2H_x^6}+\|v_1^0\|_{L_T^2H_{xy}^7}+\|\partial_t\bar{v}\|_{L_T^2H_x^6}+\|\partial_t v_1^0\|_{L_T^2H_{xy}^7}\\&+\|\partial_t^2\bar{v}\|_{L_T^2H_x^4}+\|\partial_t^2 v_1^0\|_{L_T^2H_{xy}^5}\le C.\end{aligned}$$

It is easy to verify that the compatibility conditions up to order 1 for problem (2.10) are fulfilled by r and s under assumption $(A1)$-$(A2)$. By the above estimates on $r(x,z,t)$ and $s(x,t)$, we can apply Proposition 3.3 to (2.10) and derive (3.23). Moreover, (3.24) follows from (2.10) and (3.23) by a similar argument as deriving (3.19). The proof is completed. □

Lemma 3.4. *Suppose the assumptions in Theorem 2.1 hold. Let $(u^0,\vec{v}^0)(x,y,t)$, $(v_2^{B,0},u^{B,1})(x,z,t)$ and $(u^{I,1},\vec{v}^{I,1})(x,y,t)$ be as derived in Proposition 2.1, Lemma 3.1 and Lemma 3.2, respectively. Then* (2.11) *admits a unique solution $v_2^{B,1}(x,z,t)$ on $[0,T]$ satisfying for all $l\in\mathbb{N}$ that*

$$\langle z\rangle^l\partial_t^k v_2^{B,1}\in L^\infty([0,T];H_x^{6-2k}H_z^1)\cap L^2([0,T];H_x^{6-2k}H_z^2),\quad k=0,1,2,3. \tag{3.25}$$

Moreover, it follows from (2.11) *and* (2.12) *that*

$$\langle z\rangle^l v_2^{B,1}\in L^\infty([0,T];H_x^4H_z^3),\quad \langle z\rangle^l\partial_t v_2^{B,1}\in L^\infty([0,T];H_x^2H_z^3) \tag{3.26}$$

and that

$$\langle z\rangle^l\partial_t^k u^{B,2}\in L^\infty([0,T];H_x^{6-2k}H_z^2)\cap L^2([0,T];H_x^{6-2k}H_z^3),\quad k=0,1,2,3. \tag{3.27}$$

Proof. Let φ be as defined in (3.14). Denote $\tilde{v}_2^{B,1}(x,z,t)=v_2^{B,1}(x,z,t)+\varphi(z)v_2^{I,1}(x,0,t)$. From (2.11) one deduces that

$$\begin{cases}\partial_t\tilde{v}_2^{B,1}+\bar{u}(x,t)\tilde{v}_2^{B,1}=\partial_z^2\tilde{v}_2^{B,1}+\rho,\\ \tilde{v}_2^{B,1}(x,z,0)=0,\\ \tilde{v}_2^{B,1}(x,0,t)=0,\end{cases} \tag{3.28}$$

where $\rho(x,z,t)=\partial_t v_2^{I,1}(x,0,t)\varphi(z)+\bar{u}(x,t)v_2^{I,1}(x,0,t)\varphi(z)-v_2^{I,1}(x,0,t)\partial_z^2\varphi(z)-2(v_2^0(x,0,t)+v_2^{B,0})\partial_z v_2^{B,0}+\int_z^\infty\Gamma(x,\eta,t)\,d\eta$ with $\Gamma(x,z,t)$ defined in (2.13). For $k=0,1,2,3$ and $l\in\mathbb{N}$ one has

$$\begin{aligned}\langle z\rangle^l\partial_t^k\rho=&[\langle z\rangle^l\varphi(z)\partial_t^{k+1}v_2^{I,1}(x,0,t)+\langle z\rangle^l\varphi(z)\partial_t^k(\bar{u}(x,t)v_2^{I,1}(x,0,t))-\langle z\rangle^l\partial_z^2\varphi(z)\partial_t^k v_2^{I,1}(x,0,t)]\\&-2\langle z\rangle^l\partial_t^k[(v_2^0(x,0,t)+v_2^{B,0})\partial_z v_2^{B,0}]+[\langle z\rangle^l\int_z^\infty\partial_t^k\Gamma(x,\eta,t)\,d\eta]\\:=&R_1-R_2+R_3.\end{aligned}$$

We proceed to estimate R_1, R_2 and R_3. First it follows from (3.16) and Lemma 3.2 that

$$\|\partial_t^k v_2^{I,1}(x,0,t)\|_{L_T^2H_x^{8-2k}}\le\|\partial_t^k v_2^{I,1}\|_{L_T^2H_{xy}^{9-2k}}\le C,\quad k=1,2,3,4 \tag{3.29}$$

and that $\|v_2^{I,1}(x,0,t)\|_{L_T^2H_x^6}\le\|v_2^{I,1}\|_{L_T^2H_{xy}^7}\le C$. Thus by (3.29) and a similar argument as deriving (3.18) one gets $\|R_1\|_{L_T^2H_x^{6-2k}L_z^2}\le C$. Moreover, it follows from the Sobolev embedding inequality that

$$\|R_2\|_{L_T^2H_x^{6-2k}L_z^2} \le \sum_{j=0}^{k}(\|\partial_t^j v_2^0\|_{L_T^2H_{xy}^{8-2j}}\|\langle z\rangle^l\partial_t^{k-j}\partial_z v_2^{B,0}\|_{L_T^\infty H_x^{6-2(k-j)}L_z^2} + \|\partial_t^j v_2^{B,0}\|_{L_T^2H_x^{8-2j}H_z^2}\|\langle z\rangle^l\partial_t^{k-j}\partial_z v_2^{B,0}\|_{L_T^\infty H_x^{6-2(k-j)}L_z^2}) \le C,$$

where we have used the following inequality

$$\begin{aligned}\|f(x,z,t)g(x,z,t)\|_{H_x^lL_z^2} \le & C_0\sum_{i=0}^{l}\|\partial_x^i f\|_{L_{xz}^\infty}\sum_{j=0}^{l}\|\partial_x^j g\|_{L_{xz}^2}\\ \le & C_0\sum_{i=0}^{l}\|\partial_x^i f\|_{H_{xz}^2}\sum_{j=0}^{l}\|\partial_x^i g\|_{L_{xz}^2} \le C_0\|f\|_{H_x^{l+2}H_z^2}\|g\|_{H_x^lL_z^2}\end{aligned} \tag{3.30}$$

for fixed $t>0$. By (3.16), Proposition 2.1, Lemma 3.1 and a similar argument as estimating $\|R_2\|_{L_T^2H_x^{6-2k}L_z^2}$ one derives for all $l\in\mathbb{N}$ and $k=0,1,2,3$ that

$$\|\langle z\rangle^{l+2}\partial_t^k\Gamma\|_{L_T^2H_x^{6-2k}L_z^2} \le C. \tag{3.31}$$

On the other hand, the Cauchy-Schwarz inequality entails for fixed $t\in[0,T]$ that

$$\begin{aligned}\|R_3\|^2_{H_x^{6-2k}L_z^2} \le & \int_0^\infty\Big(\langle z\rangle^l\int_z^\infty\|\partial_t^k\Gamma(x,\eta,t)\|_{H_x^{6-2k}}d\eta\Big)^2dz\\ \le & \int_0^\infty\langle z\rangle^{-2}dz\cdot\Big(\int_0^\infty\|\langle\eta\rangle^{l+1}\partial_t^k\Gamma\|_{H_x^{6-2k}}d\eta\Big)^2\\ \le & \int_0^\infty\langle z\rangle^{-2}dz\cdot\int_0^\infty\langle\eta\rangle^{-2}d\eta\cdot\int_0^\infty\|\langle\eta\rangle^{l+2}\partial_t^k\Gamma\|^2_{H_x^{6-2k}}d\eta\\ \le & C_0\|\langle z\rangle^{l+2}\partial_t^k\Gamma\|^2_{H_x^{6-2k}L_z^2},\end{aligned}$$

which, along with (3.31) gives rise to $\|R_3\|_{L_T^2H_x^{6-2k}L_z^2}\le C$. Then collecting the above estimates for R_1, R_2 and R_3 we deduce for all $l\in\mathbb{N}$ and $k=0,1,2,3$ that $\|\langle z\rangle^l\partial_t^k\rho\|_{L_T^2H_x^{6-2k}L_z^2}\le C$. It is easy to verify that ρ and $\bar{u}$ fulfill the compatibility conditions up to order 2 for problem (3.28) under assumption $(A1)$-$(A2)$. Then we apply Proposition 3.1 with $m=3$ to (3.28) to conclude that

$$\langle z\rangle^l\partial_t^k\tilde{v}_2^{B,1}\in L^\infty([0,T];H_x^{6-2k}H_z^1)\cap L^2([0,T];H_x^{6-2k}H_z^2),\quad k=0,1,2,3,$$

which, in conjunction with the definition of $\tilde{v}_2^{B,1}$ and (3.29), implies (3.25). Then (3.27) follows directly from (2.12), (3.25) and (3.31). Finally, by a similar argument used in deriving (3.19), one deduces (3.26) from (3.25), (2.11) and (3.31). The proof is finished. □

Lemma 3.5. *Suppose the assumptions in Theorem 2.1 hold. Let $\vec{v}^0(x,y,t)$, $v_2^{B,0}(x,z,t)$, $\vec{v}^{I,1}(x,y,t)$ and $u^{B,2}(x,z,t)$ be as derived in Proposition 2.1, Lemma 3.1, Lemma 3.2 and Lemma 3.4 respectively. Then* (2.14) *admits a unique solution $v_1^{B,2}(x,z,t)$ on $[0,T]$ such that for any $l\in\mathbb{N}$,*

$$\begin{aligned}&\langle z\rangle^l v_1^{B,2},\ \langle z\rangle^l\partial_z v_1^{B,2},\ \langle z\rangle^l\partial_t v_1^{B,2}\in L^\infty([0,T];H_x^3L_z^2)\cap L^2(0,T;H_x^3H_z^1);\\ &\langle z\rangle^l\partial_z\partial_t v_1^{B,2},\ \langle z\rangle^l\partial_t^2 v_1^{B,2}\in L^\infty([0,T];H_x^1L_z^2)\cap L^2([0,T];H_x^1H_z^1).\end{aligned} \tag{3.32}$$

Moreover, it follows from (2.14) *that*

$$\langle z\rangle^l v_1^{B,2}\in L^\infty([0,T];H_x^3H_z^2),\quad \langle z\rangle^l\partial_t v_1^{B,2}\in L^\infty([0,T];H_{xz}^2). \tag{3.33}$$

Proof. Let $r(x,z,t)=-\partial_x[2v_2^0(x,0,t)v_2^{B,0}+v_2^{B,0}v_2^{B,0}]+\partial_x u^{B,2}$ and $s(x,t)=-\partial_y v_1^{I,1}(x,0,t)$. To apply Proposition 3.3 to (2.14) we shall prove that r and s satisfy the assumptions of Proposition 3.3 with $m=3$. First, it is easy to verify that r and s fulfill the compatibility conditions up to order 1 for problem (2.14) under assumption $(A1)$-$(A2)$. Moreover, for any $l\in\mathbb{N}$ we deduce from (3.17) and (3.30) that

$$\begin{aligned}\|\langle z\rangle^l\partial_t r\|_{L_T^2H_x^3L_z^2}\le & C_0(\|v_2^0\|_{L_T^2H_{xy}^5}\|\langle z\rangle^l\partial_t v_2^{B,0}\|_{L_T^\infty H_x^4L_z^2}+\|\partial_t v_2^0\|_{L_T^2H_{xy}^5}\|\langle z\rangle^l v_2^{B,0}\|_{L_T^\infty H_x^4L_z^2}\\ &+\|v_2^{B,0}\|_{L_T^\infty H_x^6H_z^2}\|\langle z\rangle^l\partial_t v_2^{B,0}\|_{L_T^2H_x^4L_z^2}+\|\langle z\rangle^l\partial_t u^{B,2}\|_{L_T^2H_x^4L_z^2})\le C.\end{aligned}$$

Similarly, one derives $\|\langle z\rangle^l r\|_{L_T^2H_x^3L_z^2}+\|\langle z\rangle^l\partial_t^2 r\|_{L_T^2H_x^1L_z^2}\le C$. On the other hand, it follows from (3.16) and Lemma 3.2 that

$$\|s\|_{L_T^2H_x^3}+\|\partial_t s\|_{L_T^2H_x^3}+\|\partial_t^2 s\|_{L_T^2H_x^1}\le\|v_1^{I,1}\|_{L_T^2H_{xy}^5}+\|\partial_t v_1^{I,1}\|_{L_T^2H_{xy}^5}+\|\partial_t^2 v_2^{I,1}\|_{L_T^2H_{xy}^3}\le C.$$

Combining the above estimates for $r(x,z,t)$ and $s(x,t)$ we then apply Proposition 3.3 with $m=3$ to (2.14) and derive (3.32). By a similar argument as deriving (3.19), we get (3.33) from (2.14) and (3.32). The proof is completed. □

4. PROOF OF MAIN RESULTS

To show the convergence results in (2.15), we first approximate solutions $(u^\varepsilon,\vec{v}^\varepsilon)$ of (1.3), (1.7) with $\varepsilon>0$ by a superposition of outer and inner layer profiles derived in the previous section, and then estimate the remainders by the delicate energy method and *bootstrap argument*. In particular the approximation $(U^a,\vec{V}^a)(x,y,t)$ is defined as follows:

$$\begin{aligned}U^a(x,y,t)=&u^0(x,y,t)+\varepsilon^{1/2}u^{I,1}(x,y,t)+\varepsilon^{1/2}u^{B,1}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)\\ &+\varepsilon u^{B,2}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)-\varepsilon\varphi(y)u^{B,2}(x,0,t),\end{aligned}$$

$$\begin{aligned}\vec{V}^a(x,y,t)=&\vec{v}^0(x,y,t)+\Big(0,\ v_2^{B,0}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)\Big)+\varepsilon^{1/2}\vec{v}^{I,1}(x,y,t)\\ &+\varepsilon^{1/2}\vec{v}^{B,1}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)+\varepsilon\Big(v_1^{B,2}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big),\ 0\Big)\end{aligned}$$

and the remainder $(U^\varepsilon,\vec{V}^\varepsilon)(x,y,t)$ is as follows

$$U^\varepsilon(x,y,t):=\varepsilon^{-1/2}(u^\varepsilon-U^a)(x,y,t),\qquad \vec{V}^\varepsilon(x,y,t):=\varepsilon^{-1/2}(\vec{v}^\varepsilon-\vec{V}^a)(x,y,t),$$

where φ is defined in (3.14) and $\varepsilon\varphi(y)u^{B,2}(x,0,t)$, $\varepsilon v_1^{B,2}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)$ in the definition of U^a, $\vec{V}^a$ are used to homogenize the boundary values of U^ε and $\vec{V}^\varepsilon$. The initial-boundary problem for the remainder follows directly from (1.3), (1.7) and initial and boundary conditions in (2.5)-(2.14), and reads as

$$\begin{cases}U_t^\varepsilon=\varepsilon^{1/2}\nabla\cdot(U^\varepsilon\vec{V}^\varepsilon)+\nabla\cdot(U^\varepsilon\vec{V}^a)+\nabla\cdot(\vec{V}^\varepsilon U^a)+\Delta U^\varepsilon+\varepsilon^{-1/2}f^\varepsilon,\\ \vec{V}_t^\varepsilon=-\varepsilon^{3/2}\nabla(|\vec{V}^\varepsilon|^2)-2\varepsilon\nabla(\vec{V}^\varepsilon\cdot\vec{V}^a)+\nabla U^\varepsilon+\varepsilon\Delta\vec{V}^\varepsilon+\varepsilon^{-1/2}\vec{g}^\varepsilon,\\ (U^\varepsilon,\vec{V}^\varepsilon)(x,y,0)=(0,0),\\ (U^\varepsilon,V_2^\varepsilon)(x,0,t)=(0,0),\quad \partial_y V_1^\varepsilon(x,0,t)=0,\end{cases}\tag{4.1}$$

where

$$f^\varepsilon=\Delta U^a+\nabla\cdot(U^a\vec{V}^a)-U_t^a,\quad \vec{g}^\varepsilon=\varepsilon\Delta\vec{V}^a+\nabla U^a-\varepsilon\nabla(|\vec{V}^a|^2)-\vec{V}_t^a.\tag{4.2}$$

4.1. **Regularity estimates on U^ε and $\vec{V}^\varepsilon$.** This subsection is to prove the well-posedness of (4.1) in space $C([0,T];H^2_{xy}\times H^2_{xy})$. In particular, we derive the following result.

Proposition 4.1. *Suppose that the assumptions in Theorem 2.1 hold and that* $0<T\le T_{\max}$ *with* $T_{\max}$ *derived in Proposition 2.1. Then there is a positive constant* ε_T *decreasingly depending on* T *with* $\lim_{T\to 0}\varepsilon_T=0$ *(see Lemma 4.3) such that for any* $\varepsilon\in(0,\varepsilon_T]$*, the problem* (4.1) *admits a unique solution* $(U^\varepsilon,\vec{V}^\varepsilon)\in C([0,T];H^2_{xy}\times H^2_{xy})$ *on* $[0,T]$ *satisfying*

$$\|U^\varepsilon\|^2_{L^\infty_T L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^\infty_T L^2_{xy}}+\|\nabla U^\varepsilon\|^2_{L^\infty_T L^2_{xy}}+\varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^\infty_T L^2_{xy}}\le C\varepsilon^{1/2} \tag{4.3}$$

and

$$\varepsilon^{1/2}\|U^\varepsilon\|^2_{L^\infty_T H^2_{xy}}+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|^2_{L^\infty_T H^2_{xy}}+\varepsilon^{5/2}\|\vec{V}^\varepsilon\|^2_{L^2_T H^3_{xy}}\le C, \tag{4.4}$$

where the constant $C>0$ *is independent of* ε*, depending on* T*.*

We remark that the estimates (4.3) and (4.4) are crucial to prove our main result, Theorem 2.1. Before proceeding, we briefly introduce the additional difficulties encountered (compared to the one-dimensional case) and main ideas used in proving Proposition 4.1. When estimating the remainder $(U^\varepsilon,\vec{V}^\varepsilon)$ (cf. [24]), an L^2 uniform-in-ε estimates of $(u^\varepsilon,\vec{v}^\varepsilon)$ is used in the one dimensional case (cf. [24, Lemma 2.1]), while system (1.3), (1.7) in multi-dimensions lacks an energy-like structure to provide such L^2 uniform-in-ε estimates of ε-independence. The challenge in our analysis thus consists in deriving the estimates (4.3) and (4.4) for $(U^\varepsilon,\vec{V}^\varepsilon)$ without any uniform-in-ε a priori estimates of solutions $(u^\varepsilon,\vec{v}^\varepsilon)$. We shall achieve this by regarding $(u^\varepsilon,\vec{v}^\varepsilon)$ as a small perturbation of $(U^a,\vec{V}^a)$ and employing the bootstrap method by choosing ε small enough.

We next recall some basic facts for later use. For $G_1(x,z,t)\in H^k_xH^m_z$ with $k,m\in\mathbb{N}$ and fixed $t>0$, we have from the change of variables that

$$\Big\|\partial_y^m G_1\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)\Big\|_{H^k_xL^2_y}=\varepsilon^{\frac14-\frac m2}\|\partial_z^m G_1(x,z,t)\|_{H^k_xL^2_z}. \tag{4.5}$$

Similar arguments in deriving (3.16) entail that

$$\|G_2(x,0,t)\|^2_{H^k_x}\le C_0\sum_{j=0}^k\int_{-\infty}^{\infty}\|\partial_x^jG_2(x,z,t)\|^2_{H^1_z}dx=C_0\|G_2(x,z,t)\|^2_{H^k_xH^1_z}, \tag{4.6}$$

provided $G_2(x,z,t)\in H^k_xH^1_z$ for fixed $t>0$. Furthermore, if $G_3(x,z,t)\in H^3_xH^2_z$ one has

$$\begin{aligned}&\|G_3(x,0,t)\|_{L^\infty_x}\le C_0\|G_3(x,z,t)\|_{L^\infty_{xz}}\le C_0\|G_3(x,z,t)\|_{H^2_{xz}},\\&\|\partial_xG_3(x,0,t)\|_{L^\infty_x}\le C_0\|G_3(x,z,t)\|_{H^3_xH^2_z}.\end{aligned} \tag{4.7}$$

For $G_4(x,z,t)\in H^2_{xz}$ with fixed $t>0$, one deduces by the Sobolev embedding inequality that

$$\Big\|G_4\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)\Big\|_{L^\infty_{xy}}=\|G_4(x,z,t)\|_{L^\infty_{xz}}\le C_0\|G_4(x,z,t)\|_{H^2_{xz}}. \tag{4.8}$$

For $h_1(x,y,t)\in H^1_{xy}$ with fixed $t>0$, it follows from the Gagliardo-Nirenberg interpolation inequality that

$$\|h_1\|_{L^4_{xy}}\le C_0(\|h_1\|^{1/2}_{L^2_{xy}}\|\nabla h_1\|^{1/2}_{L^2_{xy}}+\|h_1\|_{L^2_{xy}}) \tag{4.9}$$

and

$$\|h_1\|_{L^4_{xy}}\le C_0\|h_1\|^{1/2}_{L^2_{xy}}\|\nabla h_1\|^{1/2}_{L^2_{xy}}, \tag{4.10}$$

provided further $h_1|_{y=0}=0$. For $h_2(x,y,t)\in H^2_{xy}$ one gets

$$\|h_2\|_{L^\infty_{xy}}\le C_0(\|h_2\|^{1/2}_{L^2_{xy}}\|\nabla^2h_2\|^{1/2}_{L^2_{xy}}+\|h_2\|_{L^2_{xy}}) \tag{4.11}$$

and

$$\|h_2\|_{L^\infty_{xy}} \le C_0 \|h_2\|^{1/2}_{L^2_{xy}} \|\nabla^2 h_2\|^{1/2}_{L^2_{xy}}, \tag{4.12}$$

provided $h_2|_{y=0} = 0$.

We shall prove Proposition 4.1 by the following Lemma 4.1- Lemma 4.4, where a priori estimates on the solutions $(U^\varepsilon, \vec{V}^\varepsilon)$ is derived based on the L^2 regularity on external force $f^\varepsilon(x,y,t)$ and $\vec{g}^\varepsilon(x,y,t)$. The assumption $0<\varepsilon<1$ and the results of Proposition 2.1, Lemma 3.1- Lemma 3.5 will be frequently used in the sequel without further clarification.

The estimates on f^ε and $\vec{g}^\varepsilon$ are given as follows.

Lemma 4.1. *Suppose that the assumptions in Theorem 2.1 hold. Let $0<T\le T_{\max}$ with $T_{\max}$ derived in Proposition 2.1. Then there exists a constant C independent of ε, such that*

$$\|f^\varepsilon\|_{L^\infty_T L^2_{xy}} \le C\varepsilon^{3/4}; \qquad \|\partial_t f^\varepsilon\|_{L^\infty_T L^2_{xy}} \le C\varepsilon^{3/4}.$$

Proof. First it follows from the definition of U^a, $\vec{V}^a$, f^ε, (2.3) and (2.9) that

$$\begin{aligned} f^\varepsilon =& \varepsilon^{1/2}\partial_x^2 u^{B,1} + \varepsilon^{1/2}\partial_y^2 u^{B,1} + \varepsilon\partial_x^2 u^{B,2} + \varepsilon\partial_y^2 u^{B,2} - \varepsilon\varphi(y)\partial_x^2 u^{B,2}(x,0,t) - \varepsilon u^{B,2}(x,0,t)\partial_y^2\varphi(y) \\ &+ \partial_x\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] \\ &+ \partial_y\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1}\big)\big] \\ &+ \partial_x\big[\big(u^{I,0} + \varepsilon^{1/2}u^{I,1}\big)\big(\varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] + \varepsilon\partial_x(u^{I,1}v_1^{I,1}) \\ &+ \partial_x\big[\big(\varepsilon^{1/2}u^{B,1} + \varepsilon u^{B,2}\big)\big(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] \\ &+ \partial_y\big[\big(u^{I,0} + \varepsilon^{1/2}u^{I,1}\big)\big(v_2^{B,0} + \varepsilon^{1/2}v_2^{B,1}\big)\big] + \varepsilon\partial_y(u^{I,1}v_2^{I,1}) \\ &+ \partial_y\big[\big(\varepsilon^{1/2}u^{B,1} + \varepsilon u^{B,2}\big)\big(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1}\big)\big] \\ &- \varepsilon^{1/2}\partial_t u^{B,1} - \varepsilon\partial_t u^{B,2} + \varepsilon\varphi(y)\partial_t u^{B,2}(x,0,t). \end{aligned}$$

Moreover, from the transformation $z = \frac{y}{\sqrt{\varepsilon}}$, (5.8), (5.10) and (2.6) we deduce that

$$\begin{aligned} \varepsilon^{1/2}\partial_y^2 u^{B,1} =& \varepsilon^{-1/2}\partial_z^2 u^{B,1} = -\varepsilon^{-1/2}u^{I,0}(x,0,t)\partial_z v_2^{B,0} = -u^{I,0}(x,0,t)\partial_y v_2^{B,0}, \\ \varepsilon\partial_y^2 u^{B,2} =& -\varepsilon^{1/2}u^{I,0}(x,0,t)\partial_y v_2^{B,1} - \varepsilon^{1/2}(u^{I,1}(x,0,t) + u^{B,1})\partial_y v_2^{B,0} - \partial_y u^{I,0}(x,0,t)v_2^{B,0} \\ &- \varepsilon^{1/2}\partial_y u^{B,1}(v_2^{I,0}(x,0,t) + v_2^{B,0}) - y\partial_y u^{I,0}(x,0,t)\partial_y v_2^{B,0}, \end{aligned}$$

which, substituted into the above expression for f^ε gives rise to

$$
\begin{aligned}
f^\varepsilon =&\varepsilon^{1/2}\partial_x^2 u^{B,1}+\varepsilon\partial_x^2 u^{B,2}-\varepsilon\varphi(y)\partial_x^2 u^{B,2}(x,0,t)-\varepsilon\partial_y^2\varphi(y)u^{B,2}(x,0,t)\\
&+\partial_x\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_1^{I,0}+\varepsilon^{1/2}v_1^{I,1}+\varepsilon^{1/2}v_1^{B,1}+\varepsilon v_1^{B,2}\big)\big]\\
&+\partial_y\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_2^{I,0}+v_2^{B,0}+\varepsilon^{1/2}v_2^{I,1}+\varepsilon^{1/2}v_2^{B,1}\big)\big]\\
&+\partial_x\big[\big(u^{I,0}+\varepsilon^{1/2}u^{I,1}+\varepsilon^{1/2}u^{B,1}+\varepsilon u^{B,2}\big)\big(\varepsilon^{1/2}v_1^{B,1}+\varepsilon v_1^{B,2}\big)\big]\\
&+\partial_x\big[\big(\varepsilon^{1/2}u^{B,1}+\varepsilon u^{B,2}\big)\big(v_1^{I,0}+\varepsilon^{1/2}v_1^{I,1}\big)\big]+\varepsilon\partial_x(u^{I,1}v_1^{I,1})+\varepsilon\partial_y(u^{I,1}v_2^{I,1})\\
&+\big(u^{I,0}(x,y,t)-u^{I,0}(x,0,t)-y\partial_y u^{I,0}(x,0,t)\big)\partial_y v_2^{B,0}\\
&+\big(\partial_y u^{I,0}(x,y,t)-\partial_y u^{I,0}(x,0,t)\big)v_2^{B,0}+\varepsilon^{1/2}\big(u^{I,0}(x,y,t)-u^{I,0}(x,0,t)\big)\partial_y v_2^{B,1}\\
&+\varepsilon^{1/2}\big(u^{I,1}(x,y,t)-u^{I,1}(x,0,t)\big)\partial_y v_2^{B,0}+\varepsilon^{1/2}\big(v_2^{I,0}(x,y,t)-v_2^{I,0}(x,0,t)\big)\partial_y u^{B,1}\\
&+\varepsilon^{1/2}\big[\partial_y u^{I,0}v_2^{B,1}+\partial_y u^{I,1}v_2^{B,0}+\partial_y v_2^{I,0}u^{B,1}\big]\\
&+\varepsilon\partial_y\big[u^{I,1}v_2^{B,1}+u^{B,1}\big(v_2^{I,1}+v_2^{B,1}\big)+u^{B,2}\big(v_2^{I,0}+v_2^{B,0}+\varepsilon^{1/2}v_2^{I,1}+\varepsilon^{1/2}v_2^{B,1}\big)\big]\\
&-\varepsilon^{1/2}\partial_t u^{B,1}-\varepsilon\partial_t u^{B,2}+\varepsilon\varphi(y)\partial_t u^{B,2}(x,0,t)\\
:=&\sum_{i=1}^{11}K_i,
\end{aligned}
\tag{4.13}
$$

where K_i represents the entirety of the i-th line in the above expression. We first prove $\|f^\varepsilon\|_{L_T^\infty L_{xy}^2}\le C\varepsilon^{3/4}$ by estimating each K_i ($1\le i\le 10$). Indeed, (4.5), (4.6), (4.7) and (4.8) lead to

$$
\begin{aligned}
&\|K_3\|_{L_T^\infty L_{xy}^2}\\
\le&\varepsilon\|\phi\|_{L_y^\infty}\|u^{B,2}(x,0,t)\|_{L_T^\infty L_x^\infty}\big(\|\partial_y v_2^{I,0}\|_{L_T^\infty L_{xy}^2}+\|\partial_y v_2^{B,0}\|_{L_T^\infty L_{xy}^2}+\|\partial_y v_2^{I,1}\|_{L_T^\infty L_{xy}^2}+\|\partial_y v_2^{B,1}\|_{L_T^\infty L_{xy}^2}\big)\\
&+\varepsilon\|\partial_y\phi\|_{L_y^2}\|u^{B,2}(x,0,t)\|_{L_T^\infty L_x^2}\big(\|v_2^{I,0}\|_{L_T^\infty L_{xy}^\infty}+\|v_2^{B,0}\|_{L_T^\infty L_{xy}^\infty}+\|v_2^{I,1}\|_{L_T^\infty L_{xy}^\infty}+\|v_2^{B,1}\|_{L_T^\infty L_{xy}^\infty}\big)\\
\le&C\varepsilon^{3/4}\|u^{B,2}\|_{L_T^\infty H_{xz}^2}\big(\|v_2^{I,0}\|_{L_T^\infty H_{xy}^2}+\|v_2^{B,0}\|_{L_T^\infty H_{xz}^2}+\|v_2^{I,1}\|_{L_T^\infty H_{xy}^2}+\|v_2^{B,1}\|_{L_T^\infty H_{xz}^2}\big)\\
\le&C\varepsilon^{3/4},
\end{aligned}
$$

where $0<\varepsilon<1$ has been used. Similar arguments further give the estimates for K_2, K_1 and K_{11} as follows:

$$
\begin{aligned}
\|K_2\|_{L_T^\infty L_{xy}^2}\le&C\varepsilon^{3/4}\|u^{B,2}\|_{L_T^\infty H_{xz}^2}\big(\|v_1^{I,0}\|_{L_T^\infty H_{xy}^2}+\|v_1^{B,1}\|_{L_T^\infty H_{xz}^2}+\|v_1^{I,1}\|_{L_T^\infty H_{xy}^2}+\|v_1^{B,2}\|_{L_T^\infty H_{xz}^2}\big)\\
\le&C\varepsilon^{3/4}
\end{aligned}
$$

and

$$
\begin{aligned}
\|K_1\|_{L_T^\infty L_{xy}^2}\le&\varepsilon^{3/4}\|u^{B,1}\|_{L_T^\infty H_x^2L_z^2}+\varepsilon^{5/4}\|u^{B,2}\|_{L_T^\infty H_x^2L_z^2}+C_0\varepsilon(\|u^{B,2}\|_{L_T^\infty H_x^2H_z^1}+\|u^{B,2}\|_{L_T^\infty L_x^2H_z^1})\\
\le&C\varepsilon^{3/4}
\end{aligned}
$$

and

$$
\|K_{11}\|_{L_T^\infty L_{xy}^2}\le\varepsilon^{3/4}\|\partial_t u^{B,1}\|_{L_T^\infty L_{xz}^2}+\varepsilon^{5/4}\|\partial_t u^{B,2}\|_{L_T^\infty L_{xz}^2}+C_0\varepsilon\|\varphi(y)\|_{L_y^2}\|\partial_t u^{B,2}\|_{L_T^\infty L_x^2H_z^1}\le C\varepsilon^{3/4}.
$$

By the Sobolev embedding inequality and (4.5) we have

$$\begin{aligned}\|K_5\|_{L_T^\infty L_{xy}^2} \le&\big(\|\partial_x\vec{v}^{I,0}\|_{L_T^\infty L_{xy}^\infty}+\varepsilon^{1/2}\|\partial_x\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^\infty}\big)\big(\varepsilon^{1/2}\|u^{B,1}\|_{L_T^\infty L_{xy}^2}+\varepsilon\|u^{B,2}\|_{L_T^\infty L_{xy}^2}\big)\\&+\big(\|\vec{v}^{I,0}\|_{L_T^\infty L_{xy}^\infty}+\varepsilon^{1/2}\|\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^\infty}\big)\big(\varepsilon^{1/2}\|\partial_x u^{B,1}\|_{L_T^\infty L_{xy}^2}+\varepsilon\|\partial_x u^{B,2}\|_{L_T^\infty L_{xy}^2}\big)\\&+\varepsilon\|\nabla u^{I,1}\|_{L_T^\infty L_{xy}^\infty}\|\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^2}+\varepsilon\|u^{I,1}\|_{L_T^\infty L_{xy}^\infty}\|\nabla\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^2}\\\le&C_0\big(\|\vec{v}^{I,0}\|_{L_T^\infty H_{xy}^3}+\varepsilon^{1/2}\|\vec{v}^{I,1}\|_{L_T^\infty H_{xy}^3}\big)\big(\varepsilon^{3/4}\|u^{B,1}\|_{L_T^\infty H_x^1L_z^2}+\varepsilon^{5/4}\|u^{B,2}\|_{L_T^\infty H_x^1L_z^2}\big)\\&+C_0\varepsilon\|u^{I,1}\|_{L_T^\infty H_{xy}^3}\|\vec{v}^{I,1}\|_{L_T^\infty H_{xy}^1}\\\le&C\varepsilon^{3/4}.\end{aligned}$$

To bound K_4, K_9 and K_{10}, we use (4.5), (4.8) and similar arguments as estimating K_5 and derive that

$$\begin{aligned}\|K_4\|_{L_T^\infty L_{xy}^2}\le&C_0\varepsilon^{3/4}(\|u^{I,0}\|_{L_T^\infty H_{xy}^3}+\|u^{I,1}\|_{L_T^\infty H_{xy}^3}+\|u^{B,1}\|_{L_T^\infty H_x^3H_z^2}+\|u^{B,2}\|_{L_T^\infty H_x^3H_z^2}\big)\\&\times\big(\|v_1^{B,1}\|_{L_T^\infty H_x^1L_z^2}+\|v_1^{B,2}\|_{L_T^\infty H_x^1L_z^2}\big)\\\le&C\varepsilon^{3/4}\end{aligned}$$

and

$$\begin{aligned}&\|K_9\|_{L_T^\infty L_{xy}^2}\\\le&C_0\varepsilon^{3/4}\big(\|u^{I,0}\|_{L_T^\infty H_{xy}^3}\|v_2^{B,1}\|_{L_T^\infty L_{xz}^2}+\|u^{I,1}\|_{L_T^\infty H_{xy}^3}\|v_2^{B,0}\|_{L_T^\infty L_{xz}^2}+\|\vec{v}^{I,0}\|_{L_T^\infty H_{xy}^3}\|u^{B,1}\|_{L_T^\infty L_{xz}^2}\big)\\\le&C\varepsilon^{3/4}\end{aligned}$$

and

$$\begin{aligned}&\|K_{10}\|_{L_T^\infty L_{xy}^2}\\\le&C_0\varepsilon^{3/4}\big[\|u^{I,1}\|_{L_T^\infty H_{xy}^3}\|v_2^{B,1}\|_{L_T^\infty L_x^2H_z^1}+(\|v_2^{I,1}\|_{L_T^\infty H_{xy}^3}+\|v_2^{B,1}\|_{L_T^\infty H_x^2H_z^3})\|u^{B,1}\|_{L_T^\infty L_x^2H_z^1}\big]\\&+C_0\varepsilon^{3/4}\big(\|v_2^{I,0}\|_{L_T^\infty H_{xy}^3}+\|v_2^{B,0}\|_{L_T^\infty H_x^2H_z^3}+\|v_2^{I,1}\|_{L_T^\infty H_{xy}^3}+\|v_2^{B,1}\|_{L_T^\infty H_x^2H_z^3}\big)\|u^{B,2}\|_{L_T^\infty L_x^2H_z^1}\\\le&C\varepsilon^{3/4}.\end{aligned}$$

We come to estimate K_6 by applying the change of variables $y=\varepsilon^{1/2}z$, Taylor's formula, (4.5), Theorem 2.1 and Lemma 3.1 to get

$$\begin{aligned}\|K_6\|_{L_T^\infty L_{xy}^2}=&\varepsilon\left\|\frac{u^{I,0}(x,y,t)-u^{I,0}(x,0,t)-y\partial_y u^{I,0}(x,0,t)}{y^2}\cdot z^2\partial_y v_2^{B,0}\right\|_{L_T^\infty L_{xy}^2}\\\le&\varepsilon\|\partial_y^2u^{I,0}\|_{L_T^\infty L_{xy}^\infty}\|z^2\partial_y v_2^{B,0}\|_{L_T^\infty L_{xy}^2}\\\le&C_0\varepsilon^{3/4}\|u^{I,0}\|_{L_T^\infty H_{xy}^4}\|\langle z\rangle^2\partial_z v_2^{B,0}\|_{L_T^\infty L_{xz}^2}\\\le&C\varepsilon^{3/4}.\end{aligned}$$

A similar argument as estimating K_6 leads to

$$\|K_7\|_{L_T^\infty L_{xy}^2}\le C_0\varepsilon^{3/4}\big(\|u^{I,0}\|_{L_T^\infty H_{xy}^4}\|\langle z\rangle v_2^{B,0}\|_{L_T^\infty L_{xz}^2}+\|u^{I,0}\|_{L_T^\infty H_{xy}^3}\|\langle z\rangle\partial_z v_2^{B,1}\|_{L_T^\infty L_{xz}^2}\big)\le C\varepsilon^{3/4}$$

and

$$\|K_8\|_{L_T^\infty L_{xy}^2}\le C_0\varepsilon^{3/4}\big(\|u^{I,1}\|_{L_T^\infty H_{xy}^3}\|\langle z\rangle\partial_z v_2^{B,0}\|_{L_T^\infty L_{xz}^2}+\|\vec{v}^{\,I,0}\|_{L_T^\infty H_{xy}^3}\|\langle z\rangle\partial_z u^{B,1}\|_{L_T^\infty L_{xz}^2}\big)\le C\varepsilon^{3/4}.$$

Substituting the above estimates for K_1 to K_{11} into (4.13) we conclude that $\|f^\varepsilon\|_{L_T^\infty L_{xy}^2}\le C\varepsilon^{3/4}$.

It remains to prove $\|\partial_t f^\varepsilon\|_{L^\infty_T L^2_{xy}} \le C\varepsilon^{3/4}$. To this end, we first note that with Banach spaces X,Y,Z if $\|fg\|_Z \le C_0\|f\|_X\|g\|_Y$ holds for all $f\in X$, $g\in Y$, then it follows that

$$\|\partial_t(fg)\|_Z \le \|\partial_t f\|_X\|g\|_Y + \|f\|_X\|\partial_t g\|_Y, \tag{4.14}$$

provided $\partial_t f\in X$ and $\partial_t g\in Y$. Thus from the estimates on K_3, (4.14), Proposition 2.1 and Lemma 3.1- Lemma 3.4, one deduces that

$$\begin{aligned}
&\|\partial_t K_3\|_{L^\infty_T L^2_{xy}}\\
\le& C\varepsilon^{3/4}\|u^{B,2}\|_{L^\infty_T H^2_{xz}}\big(\|\partial_t v_2^{I,0}\|_{L^\infty_T H^2_{xy}}+\|\partial_t v_2^{B,0}\|_{L^\infty_T H^2_{xz}}+\|\partial_t v_2^{I,1}\|_{L^\infty_T H^2_{xy}}+\|\partial_t v_2^{B,1}\|_{L^\infty_T H^2_{xz}}\big)\\
&+C\varepsilon^{3/4}\|\partial_t u^{B,2}\|_{L^\infty_T H^2_{xz}}\big(\|v_2^{I,0}\|_{L^\infty_T H^2_{xy}}+\|v_2^{B,0}\|_{L^\infty_T H^2_{xz}}+\|v_2^{I,1}\|_{L^\infty_T H^2_{xy}}+\|v_2^{B,1}\|_{L^\infty_T H^2_{xz}}\big)\\
\le& C\varepsilon^{3/4}.
\end{aligned}$$

Similarly it follows from (4.14) and the above estimates on K_1, K_2 and K_4 to K_{11} that

$$\|\partial_t K_i\|_{L^\infty_T L^2_{xy}} \le C\varepsilon^{3/4}, \qquad i=1,2,4,5,\cdots,11.$$

Combing the above estimates for $\partial_t K_1$ to $\partial_t K_{11}$ with (4.13) we end up with $\|\partial_t f^\varepsilon\|_{L^\infty_T L^2_{xy}} \le C\varepsilon^{3/4}$. The proof is completed. □

Lemma 4.2. *Suppose the assumptions in Theorem 2.1 hold. Let $0<T\le T_{\max}$ with $T_{\max}$ obtained in Proposition 2.1. Then there exists a positive constant C independent of ε, depending on T such that*

$$\|\vec{g}^\varepsilon\|_{L^\infty_T L^2_{xy}} \le C\varepsilon; \qquad \|\partial_t\vec{g}^\varepsilon\|_{L^\infty_T L^2_{xy}} \le C\varepsilon.$$

Proof. By the definition of $\vec{g}^\varepsilon$ in (4.2) we write its first component g_1^ε as follows:

$$\begin{aligned}
g_1^\varepsilon =&\big[\varepsilon\Delta v_1^{I,0}+\varepsilon^{3/2}\Delta v_1^{I,1}+\varepsilon^{3/2}\partial_x^2 v_1^{B,1}+\varepsilon^2\partial_x^2 v_1^{B,2}+\varepsilon^2\partial_y^2 v_1^{B,2}+\varepsilon\partial_x u^{B,2}-\varepsilon\varphi(y)\partial_x u^{B,2}(x,0,t)\big]\\
&-\big[2\varepsilon\vec{V}^a\cdot\partial_x\vec{V}^a+\varepsilon\partial_t v_1^{B,2}\big]\\
:=&M_1-M_2,
\end{aligned}$$

where the second equation of (2.3), (2.9) and the first equation of (2.10) have been used. We proceed to estimate M_1 and M_2. First (4.5) and (4.6) lead to

$$\begin{aligned}
\|M_1\|_{L^\infty_T L^2_{xy}} \le& C_0\big(\varepsilon\|\vec{v}^{I,0}\|_{L^\infty_T H^2_{xy}}+\varepsilon^{3/2}\|\vec{v}^{I,1}\|_{L^\infty_T H^2_{xy}}+\varepsilon^{7/4}\|v_1^{B,1}\|_{L^\infty_T H^2_x L^2_z}+\varepsilon^{9/4}\|v_1^{B,2}\|_{L^\infty_T H^2_x L^2_z}\\
&+\varepsilon^{5/4}\|v_1^{B,2}\|_{L^\infty_T L^2_x H^2_z}+\varepsilon\|u^{B,2}\|_{L^\infty_T H^1_x H^1_z}\big)\\
\le& C\varepsilon.
\end{aligned}$$

To bound M_2 we first estimate $\|\vec{V}^a\|_{L^\infty_T L^\infty_{xy}}$ by the Sobolev embedding inequality, (4.8) and $0<\varepsilon<1$ as follows

$$\begin{aligned}
\|\vec{V}^a\|_{L^\infty_T L^\infty_{xy}} \le& C_0\big(\|\vec{v}^{I,0}\|_{L^\infty_T H^2_{xy}}+\|v_2^{B,0}\|_{L^\infty_T H^2_{xz}}+\varepsilon^{1/2}\|\vec{v}^{I,1}\|_{L^\infty_T H^2_{xy}}\\
&+\varepsilon^{1/2}\|v_1^{B,1}\|_{L^\infty_T H^2_{xz}}+\varepsilon^{1/2}\|v_2^{B,1}\|_{L^\infty_T H^2_{xz}}+\varepsilon\|v_1^{B,2}\|_{L^\infty_T H^2_{xz}}\big)\\
\le& C.
\end{aligned}\tag{4.15}$$

Similar arguments further yield

$$\|\partial_t\vec{V}^a\|_{L^\infty_T L^\infty_{xy}},\ \|\partial_x\vec{V}^a\|_{L^\infty_T L^2_{xy}},\ \|\partial_x\partial_t\vec{V}^a\|_{L^\infty_T L^2_{xy}} \le C. \tag{4.16}$$

Thus by (4.15), (4.16) and (4.5) we obtain

$$\|M_2\|_{L^\infty_T L^2_{xy}} \le C_0\varepsilon(\|\vec{V}^a\|_{L^\infty_T L^\infty_{xy}}\|\partial_x\vec{V}^a\|_{L^\infty_T L^2_{xy}}+\|\partial_t v_1^{B,2}\|_{L^\infty_T L^2_{xy}}) \le C\varepsilon.$$

Hence from the above estimates for M_1, M_2 one derives $\|g_1^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$. By (4.14), the above estimates for M_1, M_2 and (4.16), we further derive that $\|\partial_t g_1^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$. It remains to estimate g_2^ε and $\partial_t g_2^\varepsilon$. Indeed from the definition of $\vec{g}^\varepsilon$ in (4.2) it follows that

$$
\begin{aligned}
g_2^\varepsilon =& \big[\varepsilon\Delta v_2^{I,0} + \varepsilon^{3/2}\Delta v_2^{I,1} + \varepsilon\partial_x^2 v_2^{B,0} + \varepsilon^{3/2}\partial_x^2 v_2^{B,1} - \varepsilon\partial_y\varphi(y)u^{B,2}(x,0,t)\big]\\
&+ \big[2\varepsilon(v_2^{I,0}(x,0,t) - v_2^{I,0}(x,y,t))\partial_y v_2^{B,0} - 2\varepsilon\partial_y v_2^{B,0}(\varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1})\big]\\
&- 2\varepsilon(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2})(\partial_y v_1^{I,0} + \varepsilon^{1/2}\partial_y v_1^{I,1} + \varepsilon^{1/2}\partial_y v_1^{B,1} + \varepsilon\partial_y v_1^{B,2})\\
&- 2\varepsilon(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1})(\partial_y v_2^{I,0} + \varepsilon^{1/2}\partial_y v_2^{I,1} + \varepsilon^{1/2}\partial_y v_2^{B,1})\\
:=& M_3 + M_4 - M_5 - M_6,
\end{aligned}
$$

where the second equation of (2.3), (2.9) and $F_2^0 = F_2^1 = 0$ in (5.13) have been used. First, by (4.5), (4.6) and $0 < \varepsilon < 1$ we get

$$
\begin{aligned}
\|M_3\|_{L_T^\infty L_{xy}^2} \le& C_0\varepsilon(\|\vec{v}^{\,I,0}\|_{L_T^\infty H_{xy}^2} + \|\vec{v}^{\,I,1}\|_{L_T^\infty H_{xy}^2} + \|v_2^{B,0}\|_{L_T^\infty H_x^2 L_z^2} + \|v_2^{B,1}\|_{L_T^\infty H_x^2 L_z^2} + \|u^{B,2}\|_{L_T^\infty L_x^2 H_z^1})\\
\le& C\varepsilon.
\end{aligned}
$$

By an analogous argument as estimating K_6 in the proof of Lemma 4.1 and (4.8) one deduces

$$
\begin{aligned}
&\|M_4\|_{L_T^\infty L_{xy}^2}\\
&\le C_0\varepsilon^{5/4}\|v_2^{I,0}\|_{L_T^\infty H_{xy}^3}\|\langle z\rangle v_2^{B,0}\|_{L_T^\infty L_x^2 H_z^1} + C_0\varepsilon^{5/4}\|v_2^{B,0}\|_{L_T^\infty L_x^2 H_z^1}(\|\vec{v}^{\,I,1}\|_{L_T^\infty H_{xy}^2} + \|v_2^{B,1}\|_{L_T^\infty H_{xz}^2})\\
&\le C\varepsilon^{5/4}.
\end{aligned}
$$

We then use the Cauchy-Schwarz inequality, (4.5) and (4.8) to derive

$$
\begin{aligned}
\|M_5\|_{L_T^\infty L_{xy}^2} \le& C_0\varepsilon(\|\vec{v}^{I,0}\|_{L_T^\infty H_{xy}^2} + \|\vec{v}^{I,1}\|_{L_T^\infty H_{xy}^2} + \|v_1^{B,1}\|_{L_T^\infty H_{xz}^2} + \|v_1^{B,2}\|_{L_T^\infty H_{xz}^2})\\
&\times(\|\partial_y\vec{v}^{I,0}\|_{L_T^\infty L_{xy}^2} + \|\partial_y\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^2} + \|\partial_z v_1^{B,1}\|_{L_T^\infty L_{xz}^2} + \|\partial_z v_1^{B,2}\|_{L_T^\infty L_{xz}^2})\\
\le& C\varepsilon.
\end{aligned}
$$

Moreover, $\|M_6\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$ follows from a similar argument. Now collecting the above estimates from M_3 to M_6, we conclude that $\|g_2^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$. Finally, from (4.14) and the above estimates from M_3 to M_6, one deduces that $\|\partial_t g_2^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$. The proof is completed. □

We next establish the L^2 estimates for U^ε and $\vec{V}^\varepsilon$.

Lemma 4.3. *Suppose that the assumptions in Proposition 4.1 hold. Assume further that the solution* $(U^\varepsilon, \vec{V}^\varepsilon)(x,y,t)$ *of* (4.1) *on* $[0,T]$ *satisfies*

$$\|U^\varepsilon\|_{L_T^\infty L_{xy}^2}^2 + \|\vec{V}^\varepsilon\|_{L_T^\infty L_{xy}^2}^2 < 1. \tag{4.17}$$

Then there exists a positive constant ε_T *(defined in* (4.26)*) decreasing in* T *with* $\lim_{T\to\infty}\varepsilon_T = 0$, *such that for any* $\varepsilon \in (0, \varepsilon_T]$ *the following holds true:*

$$\|U^\varepsilon\|_{L_T^\infty L_{xy}^2}^2 + \|\vec{V}^\varepsilon\|_{L_T^\infty L_{xy}^2}^2 \le C_2\varepsilon^{1/2} < \frac{1}{2}. \tag{4.18}$$

Moreover, there exists a constant C *independent of* ε *such that*

$$\|\nabla U^\varepsilon\|_{L_T^2 L_{xy}^2}^2 + \varepsilon\|\nabla\vec{V}^\varepsilon\|_{L_T^2 L_{xy}^2}^2 \le C\varepsilon^{1/2}. \tag{4.19}$$

Proof. First, it follows from a similar argument as deriving (4.15) that

$$\|U^a\|_{L_T^\infty L_{xy}^\infty} \le C, \quad \|\partial_t U^a\|_{L_T^\infty L_{xy}^\infty} \le C, \quad \|\partial_t\vec{V}^a\|_{L_T^\infty L_{xy}^\infty} \le C. \tag{4.20}$$

Thus we conclude from (4.20), (4.15), Lemma 4.1 and Lemma 4.2 that there exists a constant C_3 independent of ε, depending on T satisfying:

$$\|U^a\|^2_{L^\infty_T L^\infty_{xy}}+\|\vec{V}^a\|^2_{L^\infty_T L^\infty_{xy}}\le C_3,\qquad \|f^\varepsilon\|^2_{L^\infty_T L^2_{xy}}+\|\vec{g}^\varepsilon\|^2_{L^\infty_T L^2_{xy}}\le C_3\varepsilon^{3/2}. \tag{4.21}$$

We proceed by taking the L^2_{xy} inner products of the first and second equations of (4.1) with $2U^\varepsilon$ and $2\vec{V}^\varepsilon$ respectively, then adding the results to obtain

$$\begin{aligned}
&\frac{d}{dt}(\|U^\varepsilon\|^2_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})+2\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+2\varepsilon\|\nabla\vec{V}^\varepsilon(t)\|^2_{L^2_{xy}}\\
&\quad=2\int_0^\infty\int_{-\infty}^\infty(-\varepsilon^{1/2}U^\varepsilon\vec{V}^\varepsilon\cdot\nabla U^\varepsilon+\varepsilon^{3/2}|\vec{V}^\varepsilon|^2\nabla\cdot\vec{V}^\varepsilon)dxdy\\
&\quad+2\int_0^\infty\int_{-\infty}^\infty(-U^\varepsilon\vec{V}^a\cdot\nabla U^\varepsilon-U^a\vec{V}^\varepsilon\cdot\nabla U^\varepsilon+2\varepsilon(\vec{V}^a\cdot\vec{V}^\varepsilon)\nabla\cdot\vec{V}^\varepsilon)dxdy\\
&\quad+2\int_0^\infty\int_{-\infty}^\infty(\varepsilon^{-1/2}f^\varepsilon U^\varepsilon+\nabla U^\varepsilon\cdot\vec{V}^\varepsilon+\varepsilon^{-1/2}\vec{g}^\varepsilon\cdot\vec{V}^\varepsilon)dxdy\\
&:=I_1+I_2+I_3.
\end{aligned}$$

The estimate for I_1 follows from (4.9), (4.10) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
I_1\le&2\varepsilon^{1/2}\|U^\varepsilon\|_{L^4_{xy}}\|\vec{V}^\varepsilon\|_{L^4_{xy}}\|\nabla U^\varepsilon\|_{L^2_{xy}}+2\varepsilon^{3/2}\|\vec{V}^\varepsilon\|^2_{L^4_{xy}}\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}\\
\le&C_0\varepsilon^{1/2}\|U^\varepsilon\|^{1/2}_{L^2_{xy}}\|\nabla U^\varepsilon\|^{3/2}_{L^2_{xy}}(\|\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}+\|\vec{V}^\varepsilon\|_{L^2_{xy}})\\
&+C_0\varepsilon^{3/2}(\|\vec{V}^\varepsilon\|_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}\\
\le&\frac12\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\frac14\varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+C_0(\varepsilon^2\|U^\varepsilon\|^2_{L^2_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon)\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}\\
&+C_0(\varepsilon^2\|U^\varepsilon\|^2_{L^2_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^2_{xy}}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}\\
\le&\frac12\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\frac14\varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+2C_0\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}\\
&+C_0(\varepsilon^2\|U^\varepsilon\|^2_{L^2_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^2_{xy}}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}},
\end{aligned}\tag{4.22}$$

where in the last inequality we have used the estimates $(\varepsilon^2\|U^\varepsilon\|^2_{L^2_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon)<2$ thanks to (4.17) and the assumption $\varepsilon\in(0,1)$. Noting that (4.17) and $\varepsilon\in(0,1)$ further lead to $C_0(\varepsilon^2\|U^\varepsilon\|^2_{L^2_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^2_{xy}}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})<3C_0\varepsilon^{3/2}$. Hence by choosing ε small enough such that

$$\varepsilon<(12C_0)^{-2}, \tag{4.23}$$

one derives $3C_0\varepsilon^{3/2}<\frac14\varepsilon$ and deduces $C_0(\varepsilon^2\|U^\varepsilon\|^2_{L^2_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^2_{xy}}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})<\frac14\varepsilon$, which substituted into (4.22) gives rise to

$$I_1\le\frac12\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\frac12\varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+2C_0\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}.$$

Moreover, by the Cauchy-Schwarz inequality and (4.21), we deduce that

$$\begin{aligned}
I_2\le&\frac14\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\frac12\varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+8\|\vec{V}^a\|^2_{L^\infty_{xy}}\|U^\varepsilon\|^2_{L^2_{xy}}+8\|U^a\|^2_{L^\infty_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}+8\varepsilon\|\vec{V}^a\|^2_{L^\infty_{xy}}\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}\\
\le&\frac14\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\frac12\varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+8C_3(\|U^\varepsilon\|^2_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}).
\end{aligned}$$

It follows from the Cauchy-Schwarz inequality and (4.21) that

$$\begin{aligned} I_3 \leq & \frac{1}{4}\|\nabla U^\varepsilon\|^2_{L^2_{xy}} + \tilde{C}_0(\|U^\varepsilon\|^2_{L^2} + \|\vec{V}^\varepsilon\|^2_{L^2_{xy}}) + \varepsilon^{-1}(\|f^\varepsilon\|^2_{L^2_{xy}} + \|\vec{g}^\varepsilon\|^2_{L^2_{xy}}) \\ \leq & \frac{1}{4}\|\nabla U^\varepsilon\|^2_{L^2_{xy}} + \tilde{C}_0(\|U^\varepsilon\|^2_{L^2} + \|\vec{V}^\varepsilon\|^2_{L^2_{xy}}) + C_3\varepsilon^{1/2}, \end{aligned}$$

where the constant $\tilde{C}_0$ is independent of ε and t. Now collecting the above estimates for I_1- I_3, one gets under the assumption (4.23) that

$$\begin{aligned} & \frac{d}{dt}(\|U^\varepsilon\|^2_{L^2_{xy}} + \|\vec{V}^\varepsilon\|^2_{L^2_{xy}}) + \|\nabla U^\varepsilon\|^2_{L^2_{xy}} + \varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}} \\ \leq & (2C_0 + \tilde{C}_0 + 8C_3)(\|U^\varepsilon\|^2_{L^2_{xy}} + \|\vec{V}^\varepsilon\|^2_{L^2_{xy}}) + C_3\varepsilon^{1/2}, \end{aligned} \tag{4.24}$$

which, along with Gronwall's inequality yields

$$\|U^\varepsilon\|^2_{L^\infty_T L^2_{xy}} + \|\vec{V}^\varepsilon\|^2_{L^\infty_T L^2_{xy}} \leq C_3 T e^{(2C_0+\tilde{C}_0+8C_3)T}\varepsilon^{1/2}. \tag{4.25}$$

To fulfill the assumption (4.23) and to derive (4.18), we set

$$\varepsilon_T = \min\Big\{(12C_0)^{-2}, \Big(2C_3 T e^{(2C_0+\tilde{C}_0+8C_3)T}\Big)^{-2}, 1\Big\}. \tag{4.26}$$

Then for any $\varepsilon \in (0, \varepsilon_T]$, the estimates (4.18) immediately follows from (4.25). Finally integrating (4.24) over $[0,T]$ and using (4.18), we obtain (4.19). The proof is completed. □

The H^2 regularity estimate on U^ε and $\vec{V}^\varepsilon$ is given in the following lemma.

Lemma 4.4. *Let the assumptions in Lemma 4.3 hold. Then there exists a constant C independent of ε such that*

$$\begin{aligned} & \|\nabla U^\varepsilon\|^2_{L^\infty_T L^2_{xy}} + \varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^\infty_T L^2_{xy}} + \|\partial_t U^\varepsilon\|^2_{L^\infty_T L^2_{xy}} \\ & + \|\partial_t\vec{V}^\varepsilon\|^2_{L^\infty_T L^2_{xy}} + \|\nabla\partial_t U^\varepsilon\|^2_{L^2_T L^2_{xy}} + \varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2_T L^2_{xy}} \leq C\varepsilon^{1/2}. \end{aligned} \tag{4.27}$$

Consequently, it follows from (4.1) *that*

$$\varepsilon^{1/2}\|U^\varepsilon\|^2_{L^\infty_T H^2_{xy}} + \varepsilon^{3/2}\|\vec{V}^\varepsilon\|^2_{L^\infty_T H^2_{xy}} + \varepsilon^{5/2}\|\vec{V}^\varepsilon\|^2_{L^2_T H^3_{xy}} \leq C. \tag{4.28}$$

Proof. Taking the L^2_{xy} inner products of the first and second equation of (4.1) with $2\partial_t U^\varepsilon$ and $2\partial_t\vec{V}^\varepsilon$ respectively and using integration by parts, one derives after adding the results

$$\begin{aligned} & \frac{d}{dt}(\|\nabla U^\varepsilon\|^2_{L^2_{xy}} + \varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}) + 2\|\partial_t U^\varepsilon\|^2_{L^2_{xy}} + 2\|\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}} \\ = & 2\int_0^\infty\int_{-\infty}^\infty(-\varepsilon^{1/2}U^\varepsilon\vec{V}^\varepsilon\cdot\nabla\partial_t U^\varepsilon + \varepsilon^{3/2}|\vec{V}^\varepsilon|^2\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy \\ & + 2\int_0^\infty\int_{-\infty}^\infty(-U^\varepsilon\vec{V}^a\cdot\nabla\partial_t U^\varepsilon - U^a\vec{V}^\varepsilon\cdot\nabla\partial_t U^\varepsilon + 2\varepsilon(\vec{V}^a\cdot\vec{V}^\varepsilon)\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy \\ & + 2\int_0^\infty\int_{-\infty}^\infty(\varepsilon^{-1/2}f^\varepsilon\partial_t U^\varepsilon + \nabla U^\varepsilon\cdot\partial_t\vec{V}^\varepsilon + \varepsilon^{-1/2}\vec{g}^\varepsilon\cdot\partial_t\vec{V}^\varepsilon)dxdy \\ := & I_4 + I_5 + I_6. \end{aligned}$$

By (4.9), (4.10) and the Cauchy-Schwarz inequality we have

$$\begin{aligned}I_4 \leq & 2\varepsilon^{1/2}\|U^\varepsilon\|_{L^4_{xy}}\|\vec{V}^\varepsilon\|_{L^4_{xy}}\|\nabla\partial_t U^\varepsilon\|_{L^2_{xy}}+2\varepsilon^{3/2}\|\vec{V}^\varepsilon\|^2_{L^4_{xy}}\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}\\ \leq & C_0\varepsilon^{1/2}\|U^\varepsilon\|^{1/2}_{L^2_{xy}}\|\nabla U^\varepsilon\|^{1/2}_{L^2_{xy}}(\|\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}+\|\vec{V}^\varepsilon\|_{L^2_{xy}})\|\nabla\partial_t U^\varepsilon\|_{L^2_{xy}}\\ &+C_0\varepsilon^{3/2}(\|\vec{V}^\varepsilon\|_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}\\ \leq & \frac{1}{4}\|\nabla\partial_t U^\varepsilon\|^2_{L^2_{xy}}+\frac{1}{4}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}}+C_0(\|U^\varepsilon\|^2_{L^2_{xy}}\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^2\|\vec{V}^\varepsilon\|^4_{L^2_{xy}}).\end{aligned}$$

Moreover, a similar argument as estimating I_2 and I_3 yields:

$$I_5 \leq \frac{1}{4}\|\nabla\partial_t U^\varepsilon\|^2_{L^2_{xy}}+\frac{1}{2}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}}+C_3(\|U^\varepsilon\|^2_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^2_{xy}})$$

and

$$I_6 \leq \frac{1}{4}\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\tilde{C}_0(\|\partial_t U^\varepsilon\|^2_{L^2_{xy}}+\|\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}})+C_3\varepsilon^{1/2}.$$

We proceed by differentiating the first equation of (4.1) with respect to t, then multiplying the resulting equation with $2\partial_t U^\varepsilon$ in L^2_{xy} and using integration by parts to derive

$$\begin{aligned}\frac{d}{dt}\|\partial_t U^\varepsilon\|^2_{L^2_{xy}}+2\|\nabla\partial_t U^\varepsilon\|^2_{L^2_{xy}} =& -2\varepsilon^{1/2}\int_0^\infty\int_{-\infty}^\infty(\partial_t U^\varepsilon\vec{V}^\varepsilon+U^\varepsilon\partial_t\vec{V}^\varepsilon)\cdot\nabla\partial_t U^\varepsilon dxdy\\ &-2\int_0^\infty\int_{-\infty}^\infty\big(\partial_t(U^\varepsilon\vec{V}^a)+\partial_t(U^a\vec{V}^\varepsilon)\big)\cdot\nabla\partial_t U^\varepsilon dxdy\\ &+2\varepsilon^{-1/2}\int_0^\infty\int_{-\infty}^\infty\partial_t f^\varepsilon\partial_t U^\varepsilon dxdy\\ :=& I_7+I_8+I_9.\end{aligned}$$

The estimate for I_7 follows from (4.9), (4.10) and the Cauchy-Schwarz inequality

$$\begin{aligned}I_7 \leq & C_0\varepsilon^{1/2}\|\nabla\partial_t U^\varepsilon\|^{3/2}_{L^2_{xy}}\|\partial_t U^\varepsilon\|^{1/2}_{L^2_{xy}}(\|\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}+\|\vec{V}^\varepsilon\|_{L^2_{xy}})\\ &+C_0\varepsilon^{1/2}\|\nabla\partial_t U^\varepsilon\|_{L^2_{xy}}\|\nabla U^\varepsilon\|^{1/2}_{L^2_{xy}}\|U^\varepsilon\|^{1/2}_{L^2_{xy}}(\|\partial_t\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}\|\nabla\partial_t\vec{V}^\varepsilon\|^{1/2}_{L^2_{xy}}+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}})\\ \leq & \frac{1}{8}\|\nabla\partial_t U^\varepsilon\|^2_{L^2_{xy}}+\frac{1}{8}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}}+C_0\varepsilon^2(\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|^2_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^4_{L^2_{xy}})\|\partial_t U^\varepsilon\|^2_{L^2_{xy}}\\ &+C_0\varepsilon(\|U^\varepsilon\|^2_{L^2_{xy}}\|\nabla U^\varepsilon\|^2_{L^2_{xy}}+\|U^\varepsilon\|_{L^2_{xy}}\|\nabla U^\varepsilon\|_{L^2_{xy}})\|\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}}.\end{aligned}$$

By (4.15), (4.20) and the Cauchy-Schwarz inequality one derives

$$I_8 \leq \frac{1}{8}\|\nabla\partial_t U^\varepsilon\|^2_{L^2_{xy}}+C(\|\partial_t U^\varepsilon\|^2_{L^2_{xy}}+\|\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}})+C(\|U^\varepsilon\|^2_{L^2_{xy}}+\|\vec{V}^\varepsilon\|^2_{L^2_{xy}}).$$

The Cauchy-Schwarz inequality further leads to $I_9 \leq \|\partial_t U^\varepsilon\|^2_{L^2_{xy}}+\varepsilon^{-1}\|\partial_t f^\varepsilon\|^2_{L^2_{xy}}$. We next differentiate the second equation of (4.1) with respect to t, then take the L^2_{xy} inner product of $2\partial_t\vec{V}^\varepsilon$ with the resulting equation and use integration by parts to have

$$\begin{aligned}\frac{d}{dt}\|\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}}+2\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2_{xy}} =& 4\varepsilon^{3/2}\int_0^\infty\int_{-\infty}^\infty\vec{V}^\varepsilon\cdot\partial_t\vec{V}^\varepsilon(\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy\\ &+2\varepsilon\int_0^\infty\int_{-\infty}^\infty\partial_t(\vec{V}^\varepsilon\cdot\vec{V}^a)(\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy\\ &+2\int_0^\infty\int_{-\infty}^\infty(\nabla\partial_t U^\varepsilon\cdot\partial_t\vec{V}^\varepsilon+\varepsilon^{-1/2}\partial_t\vec{g}^\varepsilon\cdot\partial_t\vec{V}^\varepsilon)dxdy\\ :=& I_{10}+I_{11}+I_{12}.\end{aligned}$$

First, (4.9) and the Cauchy-Schwarz inequality entail that

$$
\begin{aligned}
I_{10} \leq & C_0\varepsilon^{3/2}(\|\vec{V}^\varepsilon\|_{L^2_{xy}}^{1/2}\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^{1/2}+\|\vec{V}^\varepsilon\|_{L^2_{xy}})(\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^{1/2}\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^{1/2}+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}})\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}\\
\leq & \frac{1}{8}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+C_0(\varepsilon^3\|\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\varepsilon^2\|\vec{V}^\varepsilon\|_{L^2_{xy}}\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}})\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2\\
& +C_0(\varepsilon^3\|\vec{V}^\varepsilon\|_{L^2_{xy}}^4+\varepsilon^2\|\vec{V}^\varepsilon\|_{L^2_{xy}}^2)\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2.
\end{aligned}
$$

Moreover, from (4.15) and (4.20) one gets

$$
I_{11} \leq \frac{1}{8}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+C_0(\|\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2).
$$

Finally, it follows from the Cauchy-Schwarz inequality that $I_{12}\leq\frac{1}{8}\|\nabla\partial_t U^\varepsilon\|_{L^2_{xy}}^2+\varepsilon^{-1}\|\partial_t\vec{g}^\varepsilon\|_{L^2_{xy}}^2+C_0\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2$. Collecting the above estimates for I_4-I_{12} we arrive at

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla U^\varepsilon\|_{L^2_{xy}}^2+\varepsilon\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\|\partial_t U^\varepsilon\|_{L^2_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2)\\
&\quad+\|\partial_t U^\varepsilon\|_{L^2_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\|\nabla\partial_t U^\varepsilon\|_{L^2_{xy}}^2+\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2\\
&\leq C(\varepsilon\|\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\|U^\varepsilon\|_{L^2_{xy}}^2\|\nabla U^\varepsilon\|_{L^2_{xy}}^2\\
&\quad+\|U^\varepsilon\|_{L^2_{xy}}^2+\|\vec{V}^\varepsilon\|_{L^2_{xy}}^4+1)\times(\|\partial_t U^\varepsilon\|_{L^2_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+1)\\
&\quad+C\varepsilon^{1/2}+\varepsilon^{-1}(\|\partial_t f^\varepsilon\|_{L^2_{xy}}^2+\|\partial_t\vec{g}^\varepsilon\|_{L^2_{xy}}^2),
\end{aligned}
\tag{4.29}
$$

where $0<\varepsilon<1$ has been used. On the other hand, from (4.1), Lemma 4.1 and Lemma 4.2, we have

$$
\|\partial_t U^\varepsilon(x,y,0)\|_{L^2_{xy}}^2=\varepsilon^{-1}\|f^\varepsilon(x,y,0)\|_{L^2_{xy}}^2\leq\varepsilon^{-1}\|f^\varepsilon\|_{L^\infty_T L^2_{xy}}^2\leq C\varepsilon^{1/2}
$$

and similarly $\|\partial_t\vec{V}^\varepsilon(x,y,0)\|_{L^2_{xy}}^2=\varepsilon^{-1}\|\vec{g}^\varepsilon(x,y,0)\|_{L^2_{xy}}^2\leq C\varepsilon$. Thus we can apply Gronwall's inequality and Lemma 4.1- Lemma 4.3 to (4.29) and derive (4.27). The estimate (4.28) follows immediately from the system (4.1) and (4.27). Indeed, by the second equation of (4.1) and (4.11) one deduces for fixed $t\in[0,T]$ that

$$
\begin{aligned}
\varepsilon^2\|\vec{V}^\varepsilon\|_{H^2_{xy}}^2 \leq & C_0(\varepsilon^3\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\vec{V}^\varepsilon\|_{L^\infty_{xy}}^2+\varepsilon^2\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\vec{V}^a\|_{L^\infty_{xy}}^2+\varepsilon^2\|\vec{V}^\varepsilon\|_{L^\infty_{xy}}^2\|\nabla\vec{V}^a\|_{L^2_{xy}}^2\\
& +\|U^\varepsilon\|_{H^1_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\varepsilon^{-1}\|\vec{g}^\varepsilon\|_{L^2_{xy}}^2)\\
\leq & C_0(\varepsilon^3\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\vec{V}^\varepsilon\|_{L^2_{xy}}\|\vec{V}^\varepsilon\|_{H^2_{xy}}+\varepsilon^2\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\vec{V}^a\|_{L^\infty_{xy}}^2\\
& +\varepsilon^2\|\vec{V}^\varepsilon\|_{L^2_{xy}}\|\vec{V}^\varepsilon\|_{H^2_{xy}}\|\nabla\vec{V}^a\|_{L^2_{xy}}^2+\|U^\varepsilon\|_{H^1_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\varepsilon^{-1}\|\vec{g}^\varepsilon\|_{L^2_{xy}}^2)\\
\leq & \frac{1}{2}\varepsilon^2\|\vec{V}^\varepsilon\|_{H^2_{xy}}^2+C_0(\varepsilon^4\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^4\|\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\varepsilon^2\|\nabla\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\vec{V}^a\|_{L^\infty_{xy}}^2\\
& +\varepsilon^2\|\vec{V}^\varepsilon\|_{L^2_{xy}}^2\|\nabla\vec{V}^a\|_{L^2_{xy}}^4+\|U^\varepsilon\|_{H^1_{xy}}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2_{xy}}^2+\varepsilon^{-1}\|\vec{g}^\varepsilon\|_{L^2_{xy}}^2).
\end{aligned}
$$

Subtracting $\frac{1}{2}\varepsilon^2\|\vec{V}^\varepsilon\|_{H^2_{xy}}^2$ from both sides of the above inequality, then using (4.27), (4.18), (4.15) and Lemma 4.2 one gets

$$
\varepsilon^2\|\vec{V}^\varepsilon\|_{L^\infty_T H^2_{xy}}^2\leq C\varepsilon^{1/2},
\tag{4.30}
$$

where we have also used $\|\nabla\vec{V}^a\|^2_{L^\infty_T L^2_{xy}} \le C\varepsilon^{-1/2}$, which follows from (4.5) and a similar argument in deriving (4.15). Moreover, one derives $\varepsilon\|U^\varepsilon\|^2_{L^\infty_T H^2_{xy}} + \varepsilon^3\|\vec{V}^\varepsilon\|^2_{L^2_T H^3_{xy}} \le C\varepsilon^{1/2}$ by a similar argument as deriving (4.30). The proof is completed. □

We come to prove Proposition 4.1 by the results of Lemma 4.3 and Lemma 4.4.

Proof of Proposition 4.1. First, by choosing $\varepsilon \in (0,\varepsilon_T]$ and using Lemma 4.3, Lemma 4.4 we deduce (4.3) and (4.4). Thus $(U^\varepsilon,\vec{V}^\varepsilon) \in C([0,T];H^2_{xy}\times H^2_{xy})$. The uniqueness can be proved by the method used in [67], and we omit the details for brevity. □

4.2. **Proof of Theorem 2.1 and Theorem 2.2.** We next prove Theorem 2.1 and Theorem 2.2 by the results of Proposition 4.1.

Proof of Theorem 2.1. First, by the fact that $(U^\varepsilon,\vec{V}^\varepsilon)$ uniquely solves problem (4.1) one deduces that $(u^\varepsilon,\vec{v}^\varepsilon)$ with $u^\varepsilon = \varepsilon^{1/2}U^\varepsilon + U^a$, $\vec{v}^\varepsilon = \varepsilon^{1/2}\vec{V}^\varepsilon + \vec{V}^a$ is the unique solution of (1.3), (1.7) with $\varepsilon \in (0,\varepsilon_T]$. Thus the regularity $(u^\varepsilon,\vec{v}^\varepsilon) \in C([0,T];H^2_{xy}\times H^2_{xy})$ follows from the fact that $(U^\varepsilon,\vec{V}^\varepsilon)$, $(U^a,\vec{V}^a) \in C([0,T];H^2_{xy}\times H^2_{xy})$. We next prove the curl-free property of $\vec{v}^\varepsilon$ by applying the operator "$\nabla\times$" to the second equation of (1.3) with $\varepsilon > 0$ to find

$$\begin{cases} (\nabla\times\vec{v}^\varepsilon)_t = \varepsilon\Delta(\nabla\times\vec{v}^\varepsilon), \\ (\nabla\times\vec{v}^\varepsilon)(x,y,0) = 0, \\ \nabla\times\vec{v}^\varepsilon|_{y=0} = 0, \end{cases} \tag{4.31}$$

where the assumption $\nabla\times\vec{v}_0 = 0$ and the boundary conditions (1.7) have been used. Consequently, the uniqueness on solution of (4.31) entails that $\nabla\times\vec{v}^\varepsilon = 0$. Moreover, (2.16) follows from Lemma 3.1. Then it remains to prove (2.15). By (4.11), (4.3) and (4.4) we get

$$\|\vec{V}^\varepsilon\|_{L^\infty_T L^\infty_{xy}} \le C_0\big(\|\nabla^2\vec{V}^\varepsilon\|^{1/2}_{L^\infty_T L^2_{xy}}\|\vec{V}^\varepsilon\|^{1/2}_{L^\infty_T L^2_{xy}} + \|\vec{V}^\varepsilon\|_{L^\infty_T L^2_{xy}}\big) \le C(\varepsilon^{-\frac{3}{8}}\cdot\varepsilon^{\frac{1}{8}} + \varepsilon^{\frac{1}{4}}) \le C\varepsilon^{-1/4}. \tag{4.32}$$

Similarly, it follows that

$$\|U^\varepsilon\|_{L^\infty_T L^\infty_{xy}} \le C_0\|\nabla^2 U^\varepsilon\|^{1/2}_{L^\infty_T L^2_{xy}}\|U^\varepsilon\|^{1/2}_{L^\infty_T L^2_{xy}} \le C\varepsilon^{-1/8}\cdot\varepsilon^{1/8} \le C. \tag{4.33}$$

Hence, the definition of $\vec{V}^\varepsilon$, the Sobolev embedding inequality, (4.8) and (4.32) lead to

$$\begin{aligned} &\|\vec{v}^\varepsilon(x,y,t) - \vec{v}^0(x,y,t) - \big(0, v_2^{B,0}\big)\big(x,\frac{y}{\sqrt{\varepsilon}},t\big)\|_{L^\infty_T L^\infty_{xy}} \\ \le& C_0\big(\varepsilon^{1/2}\|\vec{v}^{I,1}\|_{L^\infty_T H^2_{xy}} + \varepsilon^{1/2}\|v_1^{B,1}\|_{L^\infty_T H^2_{xz}} + \varepsilon^{1/2}\|v_2^{B,1}\|_{L^\infty_T H^2_{xz}} \\ &\quad + \varepsilon\|v_1^{B,2}\|_{L^\infty_T H^2_{xz}} + \varepsilon^{1/2}\|\vec{V}^\varepsilon\|_{L^\infty_T L^\infty_{xy}}\big) \\ \le& C\varepsilon^{1/4}. \end{aligned} \tag{4.34}$$

Similarly, by (4.33) and the definition of U^ε we have

$$\begin{aligned} &\|u^\varepsilon(x,y,t) - u^0(x,y,t)\|_{L^\infty_T L^\infty_{xy}} \\ \le& C_0\varepsilon^{1/2}\big(\|u^{I,1}\|_{L^\infty_T H^2_{xy}} + \|u^{B,1}\|_{L^\infty_T H^2_{xz}} + \|u^{B,2}\|_{L^\infty_T H^2_{xz}} + \|U^\varepsilon\|_{L^\infty_T L^\infty_{xy}}\big) \\ \le& C\varepsilon^{1/2}. \end{aligned} \tag{4.35}$$

The combination of (4.34) and (4.35) gives (2.15) and completes the proof. □

Proof of Theorem 2.2. By $(u^\varepsilon,\vec{v}^\varepsilon)$ and $(u^0,\vec{v}^0)$ we denote the solutions of problem (1.3), (1.7) obtained in Theorem 2.1 and Proposition 2.1, respectively. Let

$$\begin{aligned} c^\varepsilon(x,y,t) &= c_0(x,y)\exp\Big\{\int_0^t [-\varepsilon\nabla\cdot\vec{v}^\varepsilon+\varepsilon|\vec{v}^\varepsilon|^2-u^\varepsilon](x,y,\tau)\,d\tau\Big\},\\ c^0(x,y,t) &= c_0(x,y)\exp\Big\{-\int_0^t u^0(x,y,\tau)\,d\tau\Big\}. \end{aligned} \tag{4.36}$$

It is easy to verify that $(u^\varepsilon,c^\varepsilon)(x,y,t)$ and $(u^0,c^0)(x,y,t)$ solve (2.17) with $\varepsilon\in(0,\varepsilon_T]$ and $\varepsilon=0$, respectively. Indeed under the curl-free property $\nabla\times\vec{v}^\varepsilon(x,y,t)=0$, one has that

$$\Delta\vec{v}^\varepsilon=\nabla(\nabla\cdot\vec{v}^\varepsilon)-\nabla\times(\nabla\times\vec{v}^\varepsilon)=\nabla(\nabla\cdot\vec{v}^\varepsilon). \tag{4.37}$$

By this, a direct computation on (4.36) leads to

$$\begin{aligned} -\frac{\nabla c^\varepsilon}{c^\varepsilon} &= -\frac{\nabla c_0}{c_0}+\int_0^t[\varepsilon\nabla(\nabla\cdot\vec{v}^\varepsilon)-\varepsilon\nabla|\vec{v}^\varepsilon|^2+\nabla u^\varepsilon]\,d\tau\\ &=\vec{v}_0+\int_0^t[\varepsilon\Delta\vec{v}^\varepsilon-\varepsilon\nabla|\vec{v}^\varepsilon|^2+\nabla u^\varepsilon]\,d\tau\\ &=\vec{v}_0+\int_0^t\partial_\tau\vec{v}^\varepsilon\,d\tau\\ &=\vec{v}^\varepsilon, \end{aligned} \tag{4.38}$$

where the assumption $\vec{v}_0=-\frac{\nabla c_0}{c_0}$ in Theorem 2.2 and the second equation of (1.3) have been used. Thus, (4.38) along with the first equation of (1.3) with $\varepsilon>0$ implies that $(u^\varepsilon,c^\varepsilon)$ satisfies the first equation of (2.17). Following a similar argument, one deduces that $(u^\varepsilon,c^\varepsilon)$ solves the second equation and the initial-boundary conditions of system (2.17) by using (4.37) and the second equation of (1.3). Hence $(u^\varepsilon,c^\varepsilon)$ solves (2.17) with $\varepsilon\in(0,\varepsilon_T]$. Similarly, (u^0,c^0) solves (2.17) with $\varepsilon=0$. We further deduce that $(u^\varepsilon,c^\varepsilon)\in C([0,T];H^2_{xy}\times H^3_{xy})$ and $(u^0,c^0)\in C([0,T];H^9_{xy}\times H^{10}_{xy})$ by the regularity estimates of $(u^\varepsilon,\vec{v}^\varepsilon)$ and $(u^0,\vec{v}^0)$ in Theorem 2.1 and Proposition 2.1. The uniqueness follows from the standard method used in [67]. Finally, one derives (2.18) and (2.19) by (4.36), (2.15), (4.4) and following the arguments employed in the proof of [24, Theorem 2.2]. We omit it for brevity. □

5. Appendix

This section is devoted to the derivation of equations (2.3)-(2.14), by employing the asymptotic analysis, which has been used in [24, Appendix] to derive layer profiles in one dimension and in [25, Appendix] to determine the thickness of boundary layers. We omit the details for brevity and just sketch the procedure.

Step 1. Initial and boundary conditions. Substituting (2.1) into the initial conditions in (1.3) and following the arguments used in [25, Appendix], we have

$$\begin{aligned} u^{I,0}(x,y,0)&=u_0(x,y),\quad u^{B,0}(x,z,0)=0,\\ \vec{v}^{I,0}(x,y,0)&=\vec{v}_0(x,y),\quad \vec{v}^{B,0}(x,z,0)=0 \end{aligned} \tag{5.1}$$

and for $j\geq 1$

$$\begin{aligned} u^{I,j}(x,y,0)&=u^{B,j}(x,z,0)=0,\\ \vec{v}^{I,j}(x,y,0)&=\vec{v}^{B,j}(x,z,0)=0. \end{aligned} \tag{5.2}$$

For the boundary conditions, we insert (2.1) into (1.7) and use (2.2) to get for $j \in \mathbb{N}$ that

$$\bar{u}(x,t) = \sum_{j=0}^{\infty} \varepsilon^{j/2}\big[u^{I,j}(x,0,t) + u^{B,j}(x,0,t)\big],$$

$$\bar{v}(x,t) = \sum_{j=0}^{\infty} \varepsilon^{j/2}\big[v_2^{I,j}(x,0,t) + v_2^{B,j}(x,0,t)\big],$$

$$0 = \sum_{j=0}^{\infty} \varepsilon^{j/2}\big[\partial_y v_1^{I,j}(x,0,t) + \varepsilon^{-1/2}\partial_z v_1^{B,j}(x,0,t)\big] - \sum_{j=0}^{\infty} \varepsilon^{j/2}\partial_x\big[v_2^{I,j}(x,0,t) + v_2^{B,j}(x,0,t)\big].$$

To fulfill the above boundary conditions for all small $\varepsilon > 0$, it is required that

$$\begin{aligned} \bar{u}(x,t) &= u^{I,0}(x,0,t) + u^{B,0}(x,0,t), \\ \bar{v}(x,t) &= v_2^{I,0}(x,0,t) + v_2^{B,0}(x,0,t), \\ 0 &= \partial_z v_1^{B,0}(x,0,t), \\ \partial_x \bar{v}(x,t) &= \partial_y v_1^{I,0}(x,0,t) + \partial_z v_1^{B,1}(x,0,t) \end{aligned} \tag{5.3}$$

and for $j \geq 1$ that

$$\begin{aligned} 0 &= u^{I,j}(x,0,t) + u^{B,j}(x,0,t), \\ 0 &= v_2^{I,j}(x,0,t) + v_2^{B,j}(x,0,t), \\ 0 &= \partial_y v_1^{I,j}(x,0,t) + \partial_z v_1^{B,j+1}(x,0,t). \end{aligned} \tag{5.4}$$

Step 2. Equations for $u^{I,j}$ and $u^{B,j}$. We first substitute (2.1) without the inner layer profiles $(u^{B,j}, \vec{v}^{B,j})$ into the first equation of (1.3) to get the equations for outer layer profiles $u^{I,j}$:

$$u_t^{I,j} - \sum_{k=0}^{j} \nabla\cdot(u^{I,k}\vec{v}^{I,j-k}) = \Delta u^{I,j} \quad \text{for } j \in \mathbb{N}. \tag{5.5}$$

To find the equations for inner layer profiles $u^{B,j}$, by a similar argument in [24, Step 2, Appendix], namely inserting (2.1) into the first equation of (1.3) and subtracting (5.5) from the resulting equation then applying Taylor expansion to $u^{I,j}$, $\vec{v}^{I,j}$, we end up with

$$\sum_{j=-2}^{\infty} \varepsilon^{j/2}\tilde{G}^j(x,z,t) = 0, \tag{5.6}$$

where

$$\begin{cases} \tilde{G}^{-2} = -\,\partial_z^2 u^{B,0}, \\ \tilde{G}^{-1} = -\,u^{I,0}(x,0,t)\partial_z v_2^{B,0} - v_2^{I,0}(x,0,t)\partial_z u^{B,0} - \partial_z(u^{B,0}v_2^{B,0}) - \partial_z^2 u^{B,1}, \\ \tilde{G}^0 = \partial_t u^{B,0} - \partial_x[(u^{I,0}(x,0,t) + u^{B,0})v_1^{B,0}] - \partial_x(u^{B,0}v_1^{I,0}(x,0,t)) - u^{B,0}\partial_y v_2^{I,0}(x,0,t) \\ \qquad - (u^{I,0}(x,0,t) + u^{B,0})\partial_z v_2^{B,1} - (u^{I,1}(x,0,t) + u^{B,1})\partial_z v_2^{B,0} - \partial_y u^{I,0}(x,0,t)v_2^{B,0} \\ \qquad - \partial_z u^{B,0}(v_2^{I,1}(x,0,t) + v_2^{B,1}) - \partial_z u^{B,1}(v_2^{I,0}(x,0,t) + v_2^{B,0}) \\ \qquad - \partial_x^2 u^{B,0} - \partial_z^2 u^{B,2} - z\partial_y u^{I,0}(x,0,t)\partial_z v_2^{B,0} - z\partial_y v_2^{I,0}(x,0,t)\partial_z u^{B,0}, \\ \cdots\cdots \end{cases}$$

with $\tilde{G}^j = 0$ for $j \geq -2$. From $\tilde{G}^{-2} = 0$ we get $\partial_z^2 u^{B,0} = 0$, which upon integrations twice with respect to z over (z,∞) along with the assumption (H), yields

$$u^{B,0}(x,z,t) = 0 \quad \text{for } (x,z,t) \in \mathbb{R}\times\mathbb{R}_+\times[0,T]. \tag{5.7}$$

Furthermore, it follows from (5.7), $\tilde{G}^{-1}=0$ and the first identity of (5.3) that

$$\partial_z^2 u^{B,1} = -u^{I,0}(x,0,t)\partial_z v_2^{B,0} = -\bar{u}(x,t)\partial_z v_2^{B,0}, \tag{5.8}$$

which, upon integration over (z,∞) gives rise to

$$\partial_z u^{B,1} = -\bar{u}(x,t)v_2^{B,0}, \tag{5.9}$$

where the assumption (H) has been used.

Applying a similar procedure as deriving (5.9) by inserting (5.7) into $\tilde{G}_0=0$, we get

$$\begin{aligned}\partial_z^2 u^{B,2} =& -\partial_x(u^{I,0}(x,0,t)v_1^{B,0}) - u^{I,0}(x,0,t)\partial_z v_2^{B,1} - (u^{I,1}(x,0,t)+u^{B,1})\partial_z v_2^{B,0} \\ & -\partial_y u^{I,0}(x,0,t)v_2^{B,0} - \partial_z u^{B,1}(v_2^{I,0}(x,0,t)+v_2^{B,0}) - z\partial_y u^{I,0}(x,0,t)\partial_z v_2^{B,0}\end{aligned} \tag{5.10}$$

and then integrating the above equation with respect to z twice, we have

$$u^{B,2} = \bar{u}(x,t)\int_z^\infty v_2^{B,1}(x,\eta,t)\,d\eta - \int_z^\infty\int_\eta^\infty \Gamma(x,\xi,t)\,d\xi\,d\eta, \tag{5.11}$$

where

$$\begin{aligned}\Gamma(x,z,t) :=& \partial_x(u^{I,0}(x,0,t)v_1^{B,0}) + (u^{I,1}(x,0,t)+u^{B,1})\partial_z v_2^{B,0} \\ & + \partial_y u^{I,0}(x,0,t)v_2^{B,0} + \partial_z u^{B,1}(v_2^{I,0}(x,0,t)+v_2^{B,0}) + z\partial_y u^{I,0}(x,0,t)\partial_z v_2^{B,0}.\end{aligned}$$

Step 3. Equations for $\vec{v}^{I,j}$ and $\vec{v}^{B,j}$. Applying an analogous argument as Step 2 to the second equation of (1.3), we derive

$$\begin{cases}\vec{v}_t^{I,0} - \nabla u^{I,0} = 0, \\ \vec{v}_t^{I,1} - \nabla u^{I,1} = 0, \\ \vec{v}_t^{I,j} + 2\sum_{k=0}^{j-2}\nabla(\vec{v}^{I,k}\cdot\vec{v}^{I,j-2-k}) - \nabla u^{I,j} - \Delta\vec{v}^{I,j-2} = 0 \quad \text{for } j\geq 2\end{cases} \tag{5.12}$$

and

$$\sum_{j=-1}^{\infty} \varepsilon^{\frac{j}{2}}\vec{F}^j(x,z,t) = 0, \tag{5.13}$$

where $\vec{F}^j(x,z,t) = (F_1^j, F_2^j)(x,z,t)$ with

$$\begin{cases}F_1^{-1} =0, \\ F_1^0 =\partial_t v_1^{B,0} - \partial_x u^{B,0} - \partial_z^2 v_1^{B,0}, \\ F_1^1 =\partial_t v_1^{B,1} - \partial_x u^{B,1} - \partial_z^2 v_1^{B,1}, \\ F_1^2 =\partial_t v_1^{B,2} + \partial_x(2v_1^{I,0}(x,0,t)v_1^{B,0} + v_1^{B,0}v_1^{B,0} + 2v_2^{I,0}(x,0,t)v_2^{B,0} + v_2^{B,0}v_2^{B,0}) \\ \qquad - \partial_x u^{B,2} - \partial_x^2 v_1^{B,0} - \partial_z^2 v_1^{B,2}, \\ \quad \cdots\cdots\end{cases}$$

and

$$\begin{cases}F_2^{-1} = -\partial_z u^{B,0}, \\ F_2^0 = \partial_t v_2^{B,0} - \partial_z u^{B,1} - \partial_z^2 v_2^{B,0}, \\ F_2^1 = \partial_t v_2^{B,1} + 2(v_1^{I,0}(x,0,t)+v_1^{B,0})\partial_z v_1^{B,0} + 2(v_2^{I,0}(x,0,t)+v_2^{B,0})\partial_z v_2^{B,0} - \partial_z u^{B,2} - \partial_z^2 v_2^{B,1}, \\ \quad \cdots\cdots,\end{cases}$$

which leads to $F_1^j=0$, $F_2^j=0$ with $j\geq -1$ to guarantee that (5.13) holds true for all small $\varepsilon>0$. Finally, the initial boundary value problems (2.3)-(2.14) shown in section 2 follow directly from the results derived in Step 1- Step 3. Indeed, by (5.5) with $j=0$, (5.12), (5.1) and (5.3), we derive (2.3). From (5.13) with $j=0$, (5.7), (5.1) and (5.3) one deduces (2.5). Similarly, (2.7) is the combination of (5.9), (5.13) with $j=0$, (5.1) and (5.3) while (2.9) comes from (5.5) with $j=1$, (5.12), (5.2) and (5.4). Moreover (5.13), (5.2) and (5.4) with $j=1$ lead to (2.10). The combination of (5.10), (5.13) with $j=1$, (5.2), (5.4) and $v_1^{B,0}=0$ yields (2.11). Finally, (2.14) follows from (5.13) with $j=1$, (5.2) and (5.4).

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References

[1] R. Alexander, Y.-G. Wang, C.-J. Xu, and T. Yang. Well-posedness of the Prandtl equation in Sobolev space. *J. Amer. Math. Soc.*, 28:745–784, 2015.
[2] M. Chae, K. Choi, K. Kang, and J. Lee. Stability of planar traveling waves in a Keller-Segel equation on an infinite strip domain. *J. Differential Equations*, 265:237–279, 2018.
[3] M.A.J. Chaplain and A.M. Stuart. A model mechanism for the chemotactic response of endothelial cells to tumor angiogenesis factor. *IMA J. Math. Appl. Med.*, 10(3):149–168, 1993.
[4] L. Corrias, B. Perthame, and H. Zaag. A chemotaxis model motivated by angiogenesis. *C. R. Math. Acad. Sci. Paris*, 2:141–146, 2003.
[5] L. Corrias, B. Perthame, and H. Zaag. Global solutions of some chemotaxis and angiogenesis systems in high space dimensions. *Milan J. Math.*, 72:1–29, 2004.
[6] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. *Invent. Math.*, 141:579–614, 2000.
[7] C. Deng and T. Li. Well-posedness of a 3D parabolic-hyperbolic Keller-Segel system in the sobolev space framework. *J. Differential Equations*, 257:1311–1332, 2014.
[8] W. E. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta. Math. Sin.*, 16:207–218, 2000.
[9] L.C. Evans. *Partial Differential Equations: Graduate Studies in Mathematics*. American Mathematical Society, 2010.
[10] P.C. File. Considerations regarding the mathematical basis for Prandtl's boundary layer theory. *Arch. Ration. Mech. Anal.*, 28(3):184–216, 1968.
[11] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes Equations and Turbulence*. Oxford University Press, 2001.
[12] H. Frid and V. Shelukhin. Boundary layers for the Navier-Stokes equations of compressible fluids. *Comm. Math. Phys.*, 208:309–330, 1999.
[13] H. Frid and V. Shelukhin. Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry. *SIAM J. Math. Anal.*, 31:1144–1156, 2000.
[14] A. Gamba, D. Ambrosi, A. Coniglio, A. de Candia, S. Di Talia, E. Giraudo, G. Serini, L. Preziosi, and F. Bussolino. Percolation, morphogenesis, and burgers dynamics in blood vessels formation. *Phys. Rev. Lett.*, 90:118101, 2003.
[15] D. Gérard-Varet and E. Dormy. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.*, 23:591–609, 2010.
[16] R. Granero-Belinchón. Global solutions for a hyperbolic-parabolic system of chemotaxis. *J. Math. Anal. Appl.*, 449:872–883, 2017.

[17] R. Granero-Belinchón. On the fractional fisher information with applications to a hyperbolic-parabolic system of chemotaxis. *J. Differential Equations*, 262:3250–3283, 2017.

[18] E. Grenier and O. Guès. Boundary layers for viscous perturbations of noncharacteristic quasilinear hyperbolic problems. *J. Differential Equations*, 143(1):110–146, 1998.

[19] J. Guo, J.X. Xiao, H.J. Zhao, and C.J. Zhu. Global solutions to a hyperbolic-parabolic coupled system with large initial data. *Acta Math. Sci. Ser. B Engl. Ed*, 29:629–641, 2009.

[20] C. Hao. Global well-posedness for a multidimensional chemotaxis model in critical besov spaces. *Z. Angew Math. Phys.*, 63:825–834, 2012.

[21] H. Höfer, J.A. Sherratt, and P.K. Maini. Cellular pattern formation during Dictyostelium aggregation. *Physica D.*, 85:425–444, 1995.

[22] M.H. Holmes. *Introduction to perturbation methods*. Springer Science & Business Media, 2012.

[23] L. Hong and J. K. Hunter. Singularity formation and instability in the unsteady inviscid and viscous prandtl equations. *Communications in Mathematical Sciences*, 1:293–316, 2003.

[24] Q. Hou, C. J. Liu, Y. G. Wang, and Z. Wang. Stability of boundary layers for a viscous hyperbolic system arising from chemotaxis: one dimensional case. *SIAM J. Math. Anal.*, to appear, 2018.

[25] Q. Hou, Z. Wang, and K. Zhao. Boundary layer problem on a hyperbolic system arising from chemotaxis. *J. Differential Equations*, 261:5035–5070, 2016.

[26] S. Jiang and J.W. Zhang. On the non-resistive limit and the magnetic boundary-layer for one-dimensional compressible magnetohydrodynamics. *Nonlinearity*, 30:3587–3612, 2017.

[27] H.Y. Jin, J.Y. Li, and Z. Wang. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. *J. Differential Equations*, 255(2):193–219, 2013.

[28] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26(3):399–415, 1970.

[29] E.F. Keller and L.A. Segel. Model for chemotaxis. *J. Theor. Biol.*, 30:225–234, 1971.

[30] E.F. Keller and L.A. Segel. Traveling bands of chemotactic bacteria: A theoretical analysis. *J. Theor. Biol.*, 30:377–380, 1971.

[31] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Uralceva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1988.

[32] H.A. Levine and B.D. Sleeman. A system of reaction diffusion equations arising in the theory of reinforced random walks. *SIAM J. Appl. Math.*, 57:683–730, 1997.

[33] H.A. Levine, B.D. Sleeman, and M. Nilsen-Hamilton. A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I. the role of protease inhibitors in preventing angiogenesis. *Math. Biosci.*, 168:71–115, 2000.

[34] D. Li, T. Li, and K. Zhao. On a hyperbolic-parabolic system modeling chemotaxis. *Math. Models Methods Appl. Sci.*, 21:1631–1650, 2011.

[35] D. Li, R. Pan, and K. Zhao. Quantitative decay of a one-dimensional hybrid chemotaxis model with large data. *Nonlinearity*, 7:2181–2210, 2015.

[36] Dong Li, Tong Li, and Kun Zhao. On a hyperbolic-parabolic system modeling chemotaxis. *Math. Models Methods Appl. Sci.*, 21:1631–1650, 2011.

[37] H. Li and K. Zhao. Initial-boundary value problems for a system of hyperbolic balance laws arising from chemotaxis. *J. Differential Equations*, 258(2):302–338, 2015.

[38] J. Li, T. Li, and Z. Wang. Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity. *Math. Models Methods Appl. Sci.*, 24(14):2819–2849, 2014.

[39] T. Li, R. Pan, and K. Zhao. Global dynamics of a hyperbolic-parabolic model arising from chemotaxis. *SIAM J. Appl. Math.*, 72(1):417–443, 2012.

[40] T. Li and Z. Wang. Nonlinear stability of travelling waves to a hyperbolic-parabolic system modeling chemotaxis. *SIAM J. Appl. Math.*, 70(5):1522–1541, 2009.

[41] T. Li and Z. Wang. Nonlinear stability of large amplitude viscous shock waves of a hyperbolic-parabolic system arising in chemotaxis. *Math. Models Methods Appl. Sci.*, 20(10):1967–1998, 2010.

[42] T. Li and Z. Wang. Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis. *J. Differential Equations*, 250(3):1310–1333, 2011.

[43] T. Li and Z. Wang. Steadily propagating waves of a chemotaxis model. *Math. Biosci.*, 240(2):161–168, 2012.

[44] C. Liu, Y. Wang, and T. Yang. On the ill-posedness of the Prandtl equations in three-dimensional space. *Arch. Ration. Mech. Anal.*, 220:83–108, 2016.

[45] Y. Maekawa and A. Mazzucato. *The inviscid limit and boundary layers for Navier-Stokes flows*. In: Giga Y., Novotný A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Cham, 2018.

[46] A.J. Majda and A.L. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge University Press, 2002.

[47] V. Martinez, Z. Wang, and K. Zhao. Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology. *Indiana Univ. Math. J.* , 2017.

[48] J.D. Murray. *Mathematical Biology I: An Introduction*. Springer, Berlin, 3rd edition, 2002.

[49] R. Nossal. Boundary movement of chemotactic bacterial populations. *Math. Biosci.*, 13:397–406, 1972.

[50] O.A. Oleinik. The Prandtl system of equations in boundary layer theory. *Soviet Math. Dolk.*, 4:583–586, 1963.

[51] K.J. Painter, P.K. Maini, and H.G. Othmer. Stripe formation in juvenile pomacanthus explained by a generalized Turing mechanism with chemotaxis. *Proc. Natl. Acad. Sci.*, 96:5549–5554, 1999.

[52] K.J. Painter, P.K. Maini, and H.G. Othmer. A chemotactic model for the advance and retreat of the primitive streak in avian development. *Bull. Math. Biol.*, 62:501–525, 2000.

[53] H. Peng, H. Wen, and C. Zhu. Global well-posedness and zero diffusion limit of classical solutions to 3D conservation laws arising in chemotaxis. *Z. Angew Math. Phys.*, 65(6):1167–1188, 2014.

[54] H.Y. Peng, L.Z. Ruan, and C.J. Zhu. Convergence rates of zero diffusion limit on large amplitude solution to a conservation laws arising in chemotaxis. *Kinetic and Related Models*, 5:563–581, 2012.

[55] G.J. Petter, H.M. Byrne, D.L.S. Mcelwain, and J. Norbury. A model of wound healing and angiogenesis in soft tissue. *Math. Biosci.*, 136(1):35–63, 2003.

[56] L. Prandtl. *Über Flüssigkeitsbewegungen bei sehr kleiner Reibung*. In "Verh. Int. Math. Kongr., Heidelberg 1904", Teubner, 1905.

[57] L.G. Rebholz, D. Wang, Z. Wang, K. Zhao, and C. Zerfas. Initial boundary value problems for a system of parabolic conservation laws arising from chemotaxis in multi-dimensions. *Preprint*, 2018.

[58] F. Rousset. Characteristic boundary layers in real vanishing viscosity limits. *J. Differential Equations*, 210:25–64, 2005.

[59] M. Sammartino and R.E. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. existence for Euler and Prandtl equations. *Comm. Math. Phys.*, 192(2):433–461, 1998.

[60] M. Sammartino and R.E. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. construction of the Navier-Stokes solution. *Comm. Math. Phys.*, 192(2):463–491, 1998.

[61] H. Schlichting, K. Gersten, K. Krause, H.Jr. Oertel, and C. Mayes. *Boundary-Layer Theory*. 8th edition Springer, 2004.
[62] Y.S. Tao, L.H. Wang, and Z. Wang. Large-time behavior of a parabolic-parabolic chemotaxis model with logarithmic sensitivity in one dimension. *Discrete Contin. Dyn. Syst-Series B.*, 18:821–845, 2013.
[63] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, and R.E. Goldstein. Bacterial swimming and oxygen tranport near contact lines. *Proceedings of the National Academy of Sciences*, 102:2277–2282, 2005.
[64] R. Tyson, S.R. Lubkin, and J. Murray. Models and analysis of chemotactic bacterial patterns in a liquid medium. *J. Math. Biol.*, 266:299–304, 1999.
[65] Y.G. Wang and Z.P. Xin. Zero-viscosity limit of the linearized compressible Navier-Stokes equations with highly oscillatory forces in the half-plane. *SIAM J. Math. Anal.*, 37(4):1256–1298, 2005.
[66] Z. Wang, Z. Xiang, and P. Yu. Asymptotic dynamics on a singular chemotaxis system modeling onset of tumor angiogenesis. *J. Differential Equations*, 260:2225–2258, 2016.
[67] Z. Wang and K. Zhao. Global dynamics and diffusion limit of a one-dimensional repulsive chemotaxis model. *Comm. Pure Appl. Anal.*, 12:3027–3046, 2013.
[68] E. Weinan and B. Engquist. Blowup of solutions of the unsteady Prandtls equation. *Comm. Pure Appl. Math*, 50:1287–1293, 1997.
[69] Z. P. Xin and L. Q. Zhang. On the global existence of solutions to the Prandtl's system. *Advances in Mathematics*, 181:88–133, 2004.
[70] Z.P. Xin and T. Yanagisawa. Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane. *Comm. Pure Appl. Math.*, 52(4):479–541, 1999.
[71] L. Yao, T. Zhang, and C.J. Zhu. Boundary layers for compressible Navier-Stokes equations with density-dependent viscosity and cylindrical symmetry. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(5):677–709, 2011.
[72] M. Zhang and C.J. Zhu. Global existence of solutions to a hyperbolic-parabolic system. *Proceedings of the American Mathematical Society*, 135:1017–1027, 2007.

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