Approximation of quantum assemblages

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Abstract. In this paper, we have derived a set of projectors on a large Hilbert space which will universally work for approximating quantum assemblages with binary outputs and inputs. The dimension of the Hilbert space depends on the accuracy of the approximation.

Keywords: quantum steering, quantum realization, quantum assemblages, approximation.

1 Introduction

The quantum steering was introduced by Schrödinger [1] and reformulated by Wiseman et. al [2] in 2007. Roughly speaking, in this scenario, Alice would like to steer Bob's state by manipulating the measurements on her side. An assemblage σ is a set of unnormalized states $\{\sigma_x^a\}$ which Alice steers Bob into, with her measurement x and outcome a. Thus it is used to describe the correlation between Alice and Bob.

There is a question whether a given assemblage can be produced by quantum mechanics (has a quantum realization) in finite-dimensional Hilbert space, or at least, whether one can approximate the given assemblage by the ones coming from the finite-dimensional Hilbert space. Schrödinger [1] and later Hughston-Jozsa-Wooters [3] gave a positive answer to this question when Bob's local dimension is finite.

When Bob's local dimension is infinite, the answer is negative (see Theorem 2.2). In fact, the above-mentioned question is a reformulation of "Tsirelson's problem" [4] in the quantum steering scenario, which asks the compatibility of describing the quantum composite systems by using a single Hilbert space or by a tensor product of two Hilbert spaces [5]. This problem was solved by Navascués and Perez-Garcia with a negative answer [4]. However, it remains open in the Bell scenario [6, 7, 8, 9, 10].

However, there is just an exceptional case, namely, when the outputs and inputs of Alice are binary. Navascués et.al showed that an assemblage with binary outputs and inputs can be approximated by a sequence of assemblages that have quantum realization regardless of the dimension of Bob's local Hilbert space [11]. To approximate a given assemblage, one is required to determine a sequence of quantum states and projective measurements on Alice's side. In this paper, based on Navascués' results, we construct a set of projectors on a large Hilbert space by

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sampling their principle angles uniformly in the circle. Interestingly, those fixed projectors will universally work for all assemblages (see Theorem 3.3). Note here that the dimension of the Hilbert space which is needed for the construction depends on the accuracy of the approximation.

2 Quantum assemblages and their realizations

In this paper, we prefer to use the terminologies in C^*-algebra theory, in order to deal with the infinite dimensional case. For us, state means state in the C^*-algebraic sense, i.e, unital semidefinite functional on the algebra of observables. We will use a unital C^*-algebra $\mathcal{A}_i \subset B(H_i), i = 1, 2$ to describe the quantum system of Alice (resp. Bob). The composite system is described by the minimal tensor product of $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \subset B(H_1 \otimes H_2)$. For any fixed integers M, K, an assemblage can be defined [8] as a set of states $\sigma = \{\sigma_x^a(\cdot)\}_{x=0,\dots,M-1}^{a=0,\dots,K-1}$ on $\mathcal{A}_2$, such that the sum $\sum_a \sigma_x^a$ does not depend on x and $\sum_a \sigma_x^a(\mathbb{I}_2) = 1$ for every x, where $\mathbb{I}_2$ is the identity of the algebra $\mathcal{A}_2$.

Definition. [8]. We say that an assemblage $\sigma = \{\sigma_x^a(\cdot)\}_{x,a}$ on $\mathcal{A}_2$ has a *quantum* realization if there is a unital C^*-algebra $\mathcal{A}_1 \subset B(H_1)$ and a state ρ on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$, such that

$$\sigma_x^a(\cdot) = \rho[P_{x,a} \otimes \cdot], \tag{2.1}$$

where $\{P_{x,a}\}_{x,a}$ are projective operators of $\mathcal{A}_1$ such that $\sum_a P_{x,a} = \mathbb{I}_1$ for every x.

If $dim(H_2) < \infty$, It was noticed by Schrödinger (or called the HJW theorem [3]) that every assemblage has a quantum realization. More precisely, we have the following statement:

Theorem 2.1. [3, 8]. Suppose $dim(H_2) = d_2 < \infty$, then any assemblage $\sigma = \{\sigma_x^a(\cdot)\}_{x=0,\dots,M-1}^{a=0,\dots,K-1}$ on $\mathcal{M}_{d_2}(\mathbb{C})$ has a quantum realization. Moreover, the algebra $\mathcal{A}_1$ can be taken as $\mathcal{M}_{d_2}(\mathbb{C})$.

However, we can show that the HJW theorem doesn't hold when $dim(H_2) = \infty$. To this end, we recall the notion of dichotomic assemblages [12, 13, 14].

Definition. For given M, the dichotomic assemblage $\sigma = \{\sigma_x(\cdot)\}_{x=0,\dots,M-1}$ on $\mathcal{A}_2$ is given by

$$\sigma_x = \sigma_x^0 - \sigma_x^1,$$

where $\{\sigma_x^a\}_{x=0,\dots,M}^{a=0,1}$ is an assemblage on $\mathcal{A}_2$ for $K = 2$.

Suppose $\{\sigma_x^a\}_{x=0,\dots,M-1}^{a=0,1}$ has a quantum realization, it follows from Equation (2.1) that we have

$$\sigma_x(\cdot) = \rho[A_x \otimes \cdot], \tag{2.2}$$

where $A_x = P_{x,0} - P_{x,1}$ is a self-adjoint unitary operator of $\mathcal{A}_1$. Hence we can say $\sigma = \{\sigma_x\}_x$ has a quantum realization if Equation (2.2) holds.

Theorem 2.2. *There is a C^*-algebra $\mathcal{A}_2 \subset B(H_2)$ with $dim(H_2) = \infty$, and a dichotomic assemblage $\sigma = \{\sigma_x\}_x$ on $\mathcal{A}_2$, such that σ does not have a quantum realization.*

Proof. Let $M > 2$, and suppose D is a $d \times d$ deterministic diagonal matrix, where the diagonal terms of D are either 1 or -1 and $\mathrm{Tr}[D] = 0$. Let $U_x, x = 0, \ldots, M$ are independent Haar-random matrices in the group of unitary matrices $U(d)$. Thus $U_x, x = 0, \ldots, M$ are independent random variables of some probability space $(\Omega, \mathbb{P})$ [15].

Define the following states on $\mathcal{A}_2 := \mathcal{M}_d(L_\infty(\Omega, \mathbb{P}))$:

$$\sigma_x^a(\cdot) = \frac{\mathrm{Tr}}{d} \otimes \mathbb{E}\left[\frac{\mathbb{I}_d + (-1)^a U_x D U_x^*}{2} \cdot\right], \; a = 0, 1, x = 0, \ldots, M-1,$$

where $\mathbb{E}$ is the expectation respect to the probability space $(\Omega, \mathbb{P})$. Let

$$\sigma_x(\cdot) = \sigma_x^0(\cdot) - \sigma_x^1(\cdot) = \frac{\mathrm{Tr}}{d} \otimes \mathbb{E}[U_x D U_x^* \cdot], \; x = 0, \ldots, M-1.$$

Obviously, $H_2 = \mathbb{C}^d \otimes L_2(\Omega, \mathbb{P})$ and it is easy to see that $\sigma = \{\sigma_x\}_{x=0,\ldots,M-1}$ is a dichotomic assemblage on $\mathcal{A}_2$.

Define a set of operators $F = \{F_x\}_{x=0,\ldots,M-1} \subset \mathcal{A}_2$ as the following:

$$F_x = U_x D U_x^*, \; x = 0, \ldots, M-1.$$

It is easy to see that

$$\sum_{x=0,\ldots,M-1} \sigma_x(F_x) = \mathbb{E} \sum_{x=0,\ldots,M-1} \frac{\mathrm{Tr}[D^2]}{d} = M. \tag{2.3}$$

On the other hand, suppose that the dichotomic assemblage σ can be represented by Equation (2.2), i.e, there is a state ρ on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ and operators $A_x \in \mathcal{A}_1, x = 0, \ldots, M-1$, such that

$$\sigma_x(\cdot) = \rho[A_x \otimes \;\cdot].$$

Then we have

$$\begin{aligned} \sum_{x=0,\ldots,M-1} \sigma_x(F_x) &= \sum_{x=0,\ldots,M-1} \rho[A_x \otimes F_x] \\ &\leq \left\| \sum_{x=0,\ldots,M-1} A_x \otimes F_x \right\|, \end{aligned} \tag{2.4}$$

where $\|\cdot\|$ is the operator norm on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$. It is well known in the random matrix theory that (e.g. see [16, 17])

$$\left\| \sum_{x=0,\ldots,M-1} A_x \otimes F_x \right\| \leq 2\sqrt{M} < M$$

as $d \to \infty$. Therefore, we conclude that Equation (2.3) will be violated as $d \to \infty$, which completes our proof. □

Remark. Due to random matrix theory, we note that the local (in Bob's side) observables $F_x = U_x D U_x^*, x = 0, \ldots, M-1$ in the observable algebra $\mathcal{M}_d(\mathbb{C})$ (with a faithful state $\frac{Tr}{d} \otimes \mathbb{E}$)

can be described by random Haar unitaries $u_x, x = 0, \ldots, M-1$ in some infinite dimensional C^*-algebra $\mathcal{A}_2$ with a faithful state ϕ (thus the pair $(\mathcal{A}_2, \phi)$ is called a C^*-probability space [15]) when $d \to \infty$. Therefore, we can just assume Bob's system is described by $\mathcal{A}_2$, and write

$$\sigma_x^a(\cdot) = \phi\left[\frac{\mathbb{I} + (-1)^a u_x}{2}\cdot\right],$$

where $u_x^* = u_x, u_x^2 = \mathbb{I}$. Set $F_x = u_x$. The by repeating the argument in the above proof. We have

$$\sum_x \sigma_x(F_x) = M,$$

and if we assume σ has a quantum realization, then we have

$$\sum_x \sigma_x(F_x) \leq \left\| \sum_x A_x \otimes u_x \right\| \leq 2\sqrt{M},$$

which contradicts to the above equation.

Remark. We note that the randomness of F_x is crucial. If F_x is deterministic, the operator norm $\|\sum_x A_x \otimes F_x\|$ has a upper bounded M by the triangle inequality. On the other hand, the upper bound M can be saturated by taking $\dim(H_1) = d$, and $F_x = A_x, x = 0, \ldots, M-1$. Namely,

$$\begin{aligned}\left\| \sum_x A_x \otimes A_x \right\| &\geq \mathrm{Tr}\left[\left(\sum_x A_x \otimes A_x\right) \frac{\sum_{ij=1}^d |ii\rangle\langle jj|}{d}\right] \\ &= \frac{1}{d}\sum_x \mathrm{Tr}[\mathbb{I}_d] = M,\end{aligned}$$

where we have used the fact that $A_x^* = A_x$ and $A_x^2 = \mathbb{I}_d$.

The statement of Theorem 2.2 can be understood as a negative answer to the Tsirelson problem in the context of quantum steering. Namely, not all assemblages can be implemented by Equation (2.1). It was firstly studied by M. Navascués and D. Perez-Garcia [4] that there exists a steering protocol where the tensor and the commutation assumptions for the bipartite correlation are distinguishable. In their paper, they used $\mathcal{A}_2 = C^*_{red}(*_M \mathbb{Z}_2)$.

3 Approximation of assemblages when (M, K) = (2, 2)

It was shown by Navascués et.al [11] that the HJW theorem (asymptotically) holds when $(M, K) = (2, 2)$ regardless the dimension. Namely, the correlation between Alice and Bob can be always (approximately) realized in finite dimensional local systems. Therefore, for $(M, K) = (2, 2)$ case, it is enough to consider the assemblages on finite dimensional C^*-algebras, which of course has a quantum realization due to the HJW theorem.

In this section, we will construct a set of projectors such that all assemblages for $(M, K) = (2, 2)$ can be approximated by the assemblages realized by those fixed projectors. Our idea is motivated by [18], where they considered the approximation of quantum correlation boxes with binary inputs and outputs. Due to Masanes' results [19], they can construct a fixed set of projectors that would universally work for all quantum correlation boxes. However, there is a difference between our work and theirs. The key point of our work is the following C-S decomposition of two projectors [20]. Given a pair of d-dimensional projectors P and Q, there exists an orthonormal basis in which the two projectors are jointly block-diagonal. Moreover, the blocks can be either 1-dimensional, in which case P and Q either have a 0 or a 1 in that block, or 2-dimensional, in which case they can be written in the form

$$P = \frac{1}{2}\begin{pmatrix} 1-\cos\theta & -\sin\theta \\ -\sin\theta & 1+\cos\theta \end{pmatrix}, \; Q = \frac{1}{2}\begin{pmatrix} 1-\cos\theta & \sin\theta \\ \sin\theta & 1+\cos\theta \end{pmatrix}.$$

The angles θ are called the principal angles between the subspaces.

Definition. Define the following Hilbert space:

$$H = \bigoplus_{k=1}^{n} H^k, \tag{3.5}$$

where $H^k = \mathbb{C}^2$. And the (fixed) projectors acting on H are given by:

$$\begin{cases} Q_{0,0} = \oplus_{k=1}^{n} Q_{0,0}^k, \; Q_{0,1} = \mathbb{I} - Q_{0,0}, \\ Q_{1,0} = \oplus_{k=1}^{n} Q_{1,0}^k, \; Q_{1,1} = \mathbb{I} - Q_{1,0}, \end{cases} \tag{3.6}$$

where

$$Q_{0,0}^k = \frac{1}{2}\begin{pmatrix} 1-\cos\alpha_k & -\sin\alpha_k \\ -\sin\alpha_k & 1+\cos\alpha_k \end{pmatrix}, \; Q_{1,0}^k = \frac{1}{2}\begin{pmatrix} 1-\cos\alpha_k & \sin\alpha_k \\ \sin\alpha_k & 1+\cos\alpha_k \end{pmatrix},$$

and $\alpha_k = \frac{k\pi}{2n} \in \left[0, \frac{\pi}{2}\right]$.

Proposition 3.1. *Let $A_x = Q_{x,0} - Q_{x,1}, x = 0, 1,$ then we have the following properties:*

1. $A_x^* = A_x, A_x^2 = \mathbb{I},$ *and*
$$\mathrm{Tr}(A_x) = 0, x = 0, 1;$$
2. *For any sequence $i_1, \ldots, i_l \in \{0, 1\}$ such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{l-1} \neq i_l$, we have*
$$\lim_{n\to\infty} \frac{1}{2n} \mathrm{Tr}(A_{i_1} A_{i_2} \ldots A_{i_l}) = 0. \tag{3.7}$$

Proof. 1. It is clear that $A_x^* = A_x$ and $A_x^2 = \mathbb{I}$. Moreover,

$$\begin{aligned} \mathrm{Tr}(A_x) = \mathrm{Tr}(Q_{x,0} - Q_{x,1}) &= \sum_{k=1}^{n} \mathrm{Tr}(Q_{x,0}^k - Q_{x,1}^k) \\ &= \sum_{k=1}^{n} \mathrm{Tr}(2Q_{x,0}^k - \mathbb{I}) = 0, x = 0, 1. \end{aligned}$$

2. For Equation (3.7), it is equivalent to show that for any given integer l

$$\mathrm{Tr}((A_0A_1)^lA_0)=0,\ \lim_{n\to\infty}\frac{1}{2n}\mathrm{Tr}((A_0A_1)^l)=0,$$

and

$$\mathrm{Tr}((A_1A_0)^lA_1)=0,\ \lim_{n\to\infty}\frac{1}{2n}\mathrm{Tr}((A_1A_0)^l)=0.$$

To this end, by direct calculation we have

$$(A_0A_1)^l=\oplus_{k=1}^n\begin{pmatrix}\cos 2l\alpha_k & -\sin 2l\alpha_k\\ \sin 2l\alpha_k & \cos 2l\alpha_k\end{pmatrix}.$$

Clearly

$$(A_0A_1)^lA_0=\oplus_{k=1}^n\begin{pmatrix}0 & -\sin(2l+1)\alpha_k\\ -\sin(2l+1)\alpha_k & 0\end{pmatrix}.$$

Thus we have

$$\mathrm{Tr}((A_0A_1)^lA_0)=0.$$

Moreover,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{2n}\mathrm{Tr}((A_0A_1)^l)&=\lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^n 2\cos 2l\alpha_k\\ &=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\cos\frac{kl\pi}{n}\\ &=\int_0^\pi\cos l\theta d\theta=0.\end{aligned}$$

Similarly we can show that

$$\mathrm{Tr}((A_0A_1)^lA_0)=0,\ \lim_{n\to\infty}\frac{1}{2n}\mathrm{Tr}((A_0A_1)^l)=0.$$

□

The above proposition reveals that A_0, A_1 are free observables in the asymptotic sense (as $n\to\infty$), see [17] for the notion of free observables. Hence follows from the results of [17] we have the following direct corollary:

Corollary 3.2. *The dichotomic observables $A_x, x=0,1$ defined in the above proposition can be used to saturate the Tsirelson bound of CHSH-Bell inequality.*

Definition. For any given quantum state ρ on $B(H)\otimes\mathcal{M}_d(\mathbb{C})$, we define a quantum assemblage $\sigma_\rho=\{\sigma^a_{x,\rho}(\cdot)\}_{x,a}$ using the projectors given in Equation (3.6),

$$\sigma^a_{x,\rho}(\cdot)=\rho\left[Q_{x,a}\otimes\cdot\right],\ x,a=0,1. \tag{3.8}$$

Our main result is that for any given quantum assemblage σ on $\mathcal{M}_d(\mathbb{C})$, we can always find a state ρ on $B(H)\otimes\mathcal{M}_d(\mathbb{C})$, such that σ_ρ approximates to σ with respect to the distance $d(\cdot,\cdot)$. We define the distance $d(\cdot,\cdot)$ as follows

$$d(\sigma,\sigma_\rho) := \sum_{x,a} \|\sigma_x^a - \sigma_{x,\rho}^a\|,$$

where $\|\cdot\|$ is the norm on the dual space of $\mathcal{M}_d(\mathbb{C})$, i.e, Schatten 1 norm on $\mathcal{M}_d(\mathbb{C})$. More precisely, we have the following theorem:

Theorem 3.3. *For any quantum assemblage $\sigma = \{\sigma_x^a\}_{x,a=0,1}$ on $\mathcal{M}_d(\mathbb{C})$, there is a state ρ on $B(H)\otimes\mathcal{M}_d(\mathbb{C})$, such that*

$$\lim_{n\to\infty} d(\sigma,\sigma_\rho) = 0, \tag{3.9}$$

where H and σ_ρ are given by Equations (3.5) *and* (3.8) *respectively.*

Proof. Due to the HJW theorem, there is a quantum state ρ' on $\mathcal{M}_d(\mathbb{C})\otimes\mathcal{M}_d(\mathbb{C})$ and projective operators $P_{x,a}$ on $\mathbb{C}^d$ such that

$$\sigma_x^a(\cdot) = \rho'\left[P_{x,a}\otimes\cdot\right],\ x,a=0,1. \tag{3.10}$$

For simplicity, we can assume that d is an odd number. We apply the C-S decomposition to the pair $(P_{0,0},P_{1,0})$, and we can further assume the blocks are all 2-dimensional. Thus we have

$$\begin{cases} P_{0,0} = \oplus_{k=1}^{d/2} P_{0,0}^k,\ P_{0,1} = \mathbb{I} - P_{0,0}, \\ P_{1,0} = \oplus_{k=1}^{d/2} P_{1,0}^k,\ P_{1,1} = \mathbb{I} - P_{1,0}, \end{cases} \tag{3.11}$$

where

$$P_{0,0}^k = \frac{1}{2}\begin{pmatrix} 1-\cos\theta_k & -\sin\theta_k \\ -\sin\theta_k & 1+\cos\theta_k \end{pmatrix},\ P_{1,0}^k = \frac{1}{2}\begin{pmatrix} 1-\cos\theta_k & \sin\theta_k \\ \sin\theta_k & 1+\cos\theta_k \end{pmatrix}.$$

Note that the principle angles θ_k are fixed and only depend on the given projective measurements.

It is possible (for sufficiently large n) to find integer numbers k_0 for every k, such that the angle between θ_k and α_{k_0} is sufficiently small. Namely, we have $\theta_k - \alpha_{k_0} \le \pi/2n$ for all k. By identifying $\oplus_{k_0=1}^{d/2} H^{k_0} = \mathbb{C}^d$ we have the following estimation:

$$\mathrm{Tr}\left[P_{0,0}^k Q_{0,0}^{k_0}\right] = \cos^2\frac{\theta_k-\alpha_{k_0}}{2} \ge \cos^2\left(\frac{\pi}{4n}\right).$$

Similarly, we have

$$\mathrm{Tr}\left[P_{x,a}^k Q_{x,a}^{k_0}\right] \ge \cos^2\left(\frac{\pi}{4n}\right)$$

for all $x,a=0,1$.

Using the natural embedding $\iota : \left(\oplus_{k_0=1}^{d/2} H^{k_0}\right)\otimes\mathbb{C}^d \hookrightarrow H\otimes\mathbb{C}^d$, we can find the desired state ρ by extending ρ' naturally via the embedding ι. Thus

$$\begin{aligned} \sigma_{x,\rho}^a(\cdot) &= \rho\left[Q_{x,a}\otimes\cdot\right] \\ &= \rho'\left[Q_{x,a}|_{\oplus_{k_0} H_A^{k_0}}\otimes\cdot\right]. \end{aligned}$$

Finally, we have

$$\begin{aligned} d(\sigma,\sigma_\rho) &= \sum_{x,a} \left\|\sigma_x^a - \sigma_{x,\rho}^a\right\| \\ &= \sum_{x,a} \left\|\rho'[(P_{x,a} - Q_{x,a}|_{\oplus_{k_0} H^{k_0}}) \otimes \cdot]\right\| \\ &\leq \sum_{x,a} \sqrt{d}\left\|P_{x,a} - Q_{x,a}|_{\oplus_{k_0} H^{k_0}}\right\|_2 \\ &\leq 4d\sqrt{1-\cos^2\frac{\pi}{4n}} \to 0,\ as\ n \to \infty. \end{aligned}$$

For the first inequality, we have used the Hölder inequality and $\|\cdot\|_2$ denotes the Schatten 2 norm on $\mathcal{M}_d(\mathbb{C})$. □

Remark. Unfortunately, our construction doesn't work for $(M,K) \neq (2,2)$, since C-S decomposition doesn't hold for more than 2 projectors.

4 Conclusion

In this paper, we have shown for $(M,K) \neq (2,2)$, when the local dimension is infinity, the HJW theorem doesn't hold. Namely, there is an assemblage in infinite dimension which doesn't have a quantum realization. In fact, it is a reformulation of Tsirelson's problem in the context of the quantum steering, which was disproved by Navascués and Perez-Garcia. On the other hand, for $(M,K) = (2,2)$, we have given an elementary method to approximate the assemblages coming from the finite- dimensional Hilbert spaces. Contrary to the previous work, where the (projective) measurements depend on the assembalges, we have constructed projectors on a large Hilbert space, which will universally work for all assemblages.

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