Sharp Bilinear Decompositions of Products of Hardy Spaces and Their Dual Spaces

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(Joint work)

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§I. Motivations

Hardy Spaces $H^p(\mathbb{R}^n)$ / §I

▶ Let $p \in (0, 1]$. The **Hardy space** $H^p(\mathbb{R}^n)$ is defined to be the collection of all **Schwartz distributions** $f \in \mathcal{S}'(\mathbb{R}^n)$ such that their **quasi-norms**

$$\|f\|_{H^p(\mathbb{R}^n)} := \|f^*\|_{L^p(\mathbb{R}^n)} := \left\| \sup_{t\in(0,\infty)} (\varphi_t * f) \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\varphi_t * f(x) := \langle f, \frac{1}{t^n}\varphi(\frac{x-\cdot}{t})\rangle$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi \neq 0$.

▶ It is known that $H^p(\mathbb{R}^n)$ is independent of the choice of φ.

Campanato Spaces $\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)$ / §I

- Let $\alpha \in [0, \infty)$, $q \in [1, \infty]$ and $s \in \mathbb{Z}_+$ such that $s \geq \lfloor n\alpha \rfloor$. The **Campanato space** $\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)$ is defined to be the set of all locally integrable functions g such that

$$\|g\|_{\mathfrak{C}_{\alpha,q,s}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^\alpha} \left\{ \frac{1}{|B|} \int_B |g(x) - P_{B,s}(g)(x)|^q \, dx \right\}^{1/q} < \infty,$$

where $P_{B,s}g$ denotes the **minimizing polynomial** of g on B with degree $\leq s$.

- $P_{B,s}g$ is **minimizing**: for any polynomial Q with degree $\leq s$,

$$\int_B [g - P_{B,s}g] Q = 0.$$

Hardy Spaces and Their Dual Spaces / §I

- It is well known:

  - For any $p \in (0, 1]$, $q \in [1, \infty)$ and $s \in \mathbb{Z}_+ \cap [\lfloor n(\frac{1}{p} - 1) \rfloor, \infty)$,

$$(H^p(\mathbb{R}^n))^* = \mathfrak{C}_{1/p-1,q,s}(\mathbb{R}^n) =: \mathfrak{C}_{1/p-1}(\mathbb{R}^n).$$

  - If $\alpha = 0$, then $\mathfrak{C}_0(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$.

  - If $\alpha \in (0, \frac{1}{n})$, then $\mathfrak{C}_\alpha(\mathbb{R}^n) = \dot{\Lambda}_{n\alpha}(\mathbb{R}^n)$ with the **homogeneous Lipschitz norm**

$$\|g\|_{\dot{\Lambda}_{n\alpha}(\mathbb{R}^n)} := \sup_{x,\, y \in \mathbb{R}^n,\, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^{n\alpha}}.$$

Questions / §I

- For any $p \in (0,1]$ and $\alpha = 1/p - 1$, find the **smallest** linear vector space $\mathcal{Y}$ so that $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ has the following **bilinear decomposition**:

$$H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + \mathcal{Y},$$

  namely, $\exists\, S: \ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $\exists\, T: \ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to \mathcal{Y}$, which are **bilinear** and **bounded**, such that

$$f \times g = S(f,g) + T(f,g), \forall\,(f,g) \in H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n).$$

- It is well known that

$$H^1(\mathbb{R}^n) \times \ \mathrm{BMO}(\mathbb{R}^n) \not\subset L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

- **What can we do? Why are they important?**

Jacobian / §I

- **S. Müller**, Higher integrability of determinants and weak convergence in L^1, **J. Reine Angew. Math. 412 (1990), 20-34**.
- **R. R. Coifman & L. Grafakos**, Hardy space estimates for multilinear operators. I, **Rev. Mat. Iberoamericana 8 (1992), 45-67**.
- **L. Grafakos**, Hardy space estimates for multilinear operators. II, **Rev. Mat. Iberoamericana 8 (1992), 69-92**.
- **L. Grafakos**, H^1 boundedness of determinants of vector fields, **Proc. Amer. Math. Soc. 125 (1997), 3279-3288**.

[Bijz07] **A. Bonami, T. Iwaniec, P. Jones & M. Zinsmeister**, On the product of functions in BMO and H^1, **Ann. Inst. Fourier (Grenoble) 57 (2007), 1405-1439**.

Commutators / §I

- **L. D. Ky**, Bilinear decompositions and commutators of singular integral operators, **Trans. Amer. Math. Soc. 365 (2013), 2931-2958**.
- ......

Div-Curl Lemma (1) / §I

- **Div-Curl lemma**. Let $\mathbf{F} := (F_1, \ldots, F_n) \in \mathbb{X}$ with $\operatorname{curl} \mathbf{F} \equiv 0$ and $\mathbf{G} := (G_1, \ldots, G_n) \in \mathbb{Y}$ with $\operatorname{div} \mathbf{G} \equiv 0$.
Here, for any $i, j \in \{1, \ldots, n\}$,

$$\operatorname{curl} \mathbf{F} := \left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j} \right)_{i,j} = (\mathrm{D}\, \mathbf{F})^T - (\mathrm{D}\, \mathbf{F})$$

and

$$\operatorname{div} \mathbf{G} := \sum_{j=1}^{n} \frac{\partial G_j}{\partial x_j}.$$

Find **suitable function space** $\mathbb{Z}$ such that

$$\left\| \mathbf{F} \cdot \mathbf{G} := \sum_{j=1}^{n} F_j \times G_j \right\|_{\mathbb{Z}} \lesssim \|F\|_{\mathbb{X}} \|G\|_{\mathbb{Y}}.$$

Div-Curl Lemma (2) / §I

- If $p,\, q \in (\frac{n}{n+1},\, \infty)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < \frac{n+1}{n}$, then

$$\|\mathbf{F} \cdot \mathbf{G}\|_{H^r(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n)} \|\mathbf{G}\|_{H^q(\mathbb{R}^n)}.$$

- **R. R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes**, Compensated compactness and Hardy spaces, **J. Math. Pures Appl. (9) 72 (1993), 247-286**.

Div-Curl Lemma (3) / §I

- The endpoint case $r = \frac{n}{n+1}$: If $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} = \frac{n+1}{n}$, then

$$\|\mathbf{F} \cdot \mathbf{G}\|_{H^{r,\infty}(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^p(\mathbb{R}^n)} \|\mathbf{G}\|_{H^q(\mathbb{R}^n)},$$

where $H^{r,\infty}(\mathbb{R}^n)$ denotes the **weak Hardy space**.

- **T. Miyakawa**, Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations, **Kyushu J. Math. 50 (1996), 1-64**.

Div-Curl Lemma (4): Endpoint Case $q = \infty$ (1) / §I

- If $p = 1$ and $q = \infty$, then

$$\|\mathbf{F} \cdot \mathbf{G}\|_{H^{\log}(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{H^1(\mathbb{R}^n)} \|\mathbf{G}\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

  where $H^{\log}(\mathbb{R}^n)$ denotes the **Musielak-Orlicz-Hardy space** related to

$$\theta(x, t) := \frac{t}{\log(e + t) + \log(e + |x|)}$$

  (see **[K14]**).

- **Theorem** (**[Bgk12]**). The product space $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ has the following **sharp bilinear decomposition**:

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n).$$

Div-Curl Lemma (4): Endpoint Case $q = \infty$ (2) / §I

[K14] **L. D. Ky**, New Hardy spaces of Musielak-Orlicz type and boundedness of subilinear operators, **Integral Equations Operator Theory 78 (2014), 115-150**.

[Bgk12] **A. Bonami, S. Grellier and L. D. Ky**, Paraproducts and products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets, **J. Math. Pures Appl. (9) 97 (2012), 230-241**.

How about the case $p \in (0,\ 1)$?

[Bck17] **A. Bonami, J. Cao, L. D. Ky, L. Liu, D. Yang and W. Yuan**, A complete solution to bilinear decompositions of products of Hardy and Campanato spaces, **In Progress**.

Difficulties / §I

- **Restriction** from the **method of wavelets**;
- There exist more **complicated structures** of the space $\mathfrak{C}_\alpha(\mathbb{R}^n)$.

► **Theorem** (**[Bck17]**) Let $\alpha \in (0, \infty)$. Then, for any $g \in \mathfrak{C}_\alpha(\mathbb{R}^n)$ & ball $B := B(c_B, r_B)$ of $\mathbb{R}^n$, with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$,

$$\sup_{x \in B} \left[|g(x)| + \left|P_{B,s} g(x)\right|\right] \lesssim \begin{cases} [1 + |c_B| + r_B]^{n\alpha} \|g\|_{C_\alpha^+(\mathbb{R}^n)} & \text{if } n\alpha \notin \mathbb{N}, \\ [1 + |c_B| + r_B]^{n\alpha} \log(e + |c_B| + r_B) \|g\|_{C_\alpha^+(\mathbb{R}^n)} & \text{if } n\alpha \in \mathbb{N}, \end{cases}$$

where

$$\|g\|_{\mathfrak{C}_\alpha^+(\mathbb{R}^n)} := \|g\|_{\mathfrak{C}_\alpha(\mathbb{R}^n)} + \frac{1}{|B(\vec{0}_n, 1)|} \int_{B(\vec{0}_n, 1)} |g(x)| \, dx.$$

§II. Bilinear Decompositions of Products of **Hardy** and **Campanato Spaces**

Definition of Products: Case $p = 1$ / §II

- Let $f \in H^1(\mathbb{R}^n)$ and $g \in \mathrm{BMO}(\mathbb{R}^n)$. The **product** $f \times g$ is defined to be a **Schwartz distribution** in $\mathcal{S}'(\mathbb{R}^n)$ such that, for any **Schwartz function** $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle f \times g, \phi \rangle := \langle \phi g, f \rangle,$$

  where the last bracket denotes the **dual pair** between $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$. [Recall $(H^1(\mathbb{R}^n))^* = \mathrm{BMO}(\mathbb{R}^n)$.]

  - **[Bijz07]** Bonami, Iwaniec, Jones and Zinsmeister, 2007.

Theorem (**[Ny85]**) Every $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a pointwise multiplier of $\mathrm{BMO}(\mathbb{R}^n)$.

- The above product can be extended as a **distribution** on the **class of all pointwise multipliers of** $\mathrm{BMO}(\mathbb{R}^n)$.

Pointwise Multipliers on $\mathrm{BMO}(\mathbb{R}^n)$ / §II

Theorem (**[Ny85]**) A locally integrable function g is a **pointwise multiplier** of $\mathrm{BMO}(\mathbb{R}^n)$ **iff** $g \in \mathrm{BMO}_{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $f \in \mathrm{BMO}_{\log}(\mathbb{R}^n)$ **iff**

$$\|f\|_{\mathrm{BMO}_{\log}(\mathbb{R}^n)} := \sup_{B(a,r)} \frac{|\log r| + \log(e+|a|)}{|B(a,r)|} \int_{B(a,r)} |f(x) - f_{B(a,r)}|\,dx < \infty$$

with $f_{B(a,r)} := \frac{1}{|B(a,r)|}\int_{B(a,r)} f$, here $a \in \mathbb{R}^n$ and $r \in (0,\infty)$.

[Ny85] **E. Nakai and K. Yabuta**, Pointwise multipliers for functions of bounded mean oscillation, **J. Math. Soc. Japan 37 (1985), 207-218**.

Duality Between $H^{\log}(\mathbb{R}^n)$ & $\mathrm{BMO}_{\log}(\mathbb{R}^n)$ / §II

Theorem ([K14]) $(H^{\log}(\mathbb{R}^n))^* = \mathrm{BMO}_{\log}(\mathbb{R}^n)$.

- $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$ **([Bck17])**
- $(L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n))^* = L^\infty(\mathbb{R}^n) \cap \mathrm{BMO}_{\log}(\mathbb{R}^n)$ **(Sharp)**.
- $\langle f \times g, \phi\rangle := \langle \phi g, f\rangle$

- **Sharp**: Assume that $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + \mathcal{Y}$ and $\mathcal{Y}$ is smallest. Then

$$\begin{aligned} &L^\infty(\mathbb{R}^n) \cap \mathrm{BMO}_{\log}(\mathbb{R}^n) \\ &\quad = (L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n))^* \\ &\quad \subset (L^1(\mathbb{R}^n) + \mathcal{Y})^* \\ &\quad = \textbf{all pointwise multipliers of } \mathrm{BMO}(\mathbb{R}^n) \\ &\quad = L^\infty(\mathbb{R}^n) \cap \mathrm{BMO}_{\log}(\mathbb{R}^n). \end{aligned}$$

([Ny85])

Bake to Case $p \in (0,\, 1)$ / §II

Let $\alpha := 1/p - 1$**. Find a suitable function space** $\mathbb{X}$ such that

- $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + \mathbb{X}$.
- $(L^1(\mathbb{R}^n) + \mathbb{X})^* = L^\infty(\mathbb{R}^n) \cap (\mathbb{X})^*$ **characterizes the class of all pointwise multipliers of** $\mathfrak{C}_\alpha(\mathbb{R}^n)$.

The space $\mathbb{X}$ turns out to be the **Musielak-Orlicz Hardy space** $H^{\Phi_p}(\mathbb{R}^n)$ associated with the **Musielak-Orlicz growth function** Φ_p defined by setting, for any $x \in \mathbb{R}^n$ and $t \in [0,\, \infty)$,

$$\Phi_p(x,\, t) := \begin{cases} \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}} & \text{if } n\alpha \notin \mathbb{Z}_+, \\ \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{if } n\alpha \in \mathbb{Z}_+. \end{cases}$$

- $\Phi_1(x,\, t) = \theta(x,\, t)$.

M-O Lebesgue & Hardy Spaces Associated with Φ_p / §II

- Let $p \in (0, 1]$. The **Musielak-Orlicz Lebesgue space** $L^{\Phi_p}(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\Phi_p}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi_p(x, |f(x)|/\lambda)\, dx \le 1\right\} < \infty.$$

- The **Musielak-Orlicz Hardy space** $H^{\Phi_p}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that f^* belongs to $L^{\Phi_p}(\mathbb{R}^n)$ and is equipped with the quasi-norm

$$\|f\|_{H^{\Phi_p}(\mathbb{R}^n)} := \|f^*\|_{L^{\Phi_p}(\mathbb{R}^n)},$$

where $f^*(x) := \sup_{t\in(0,\infty)} |\langle f, \frac{1}{t^n}\varphi(\frac{x-\cdot}{t})\rangle|$ for any $x \in \mathbb{R}^n$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi \neq 0$.

M-O Campanato Spaces Associated with Φ_p / §II

- Let $p \in (0, 1]$ and $s \in \mathbb{Z}_+ \cap [\lfloor n(1/p-1) \rfloor, \infty)$. The **Musielak-Orlicz Campanato space** $\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ consists of all measurable functions g on $\mathbb{R}^n$ such that

$$\|g\|_{\mathfrak{C}_{\Phi_p}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)}} \int_B \left|g(x) - P_{B,s}(g)(x)\right| dx < \infty,$$

where $P_{B,s}g$ denotes the **minimizing polynomial** of g on B with degree $\leq s$.

Pointwise Multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$/ §I

Theorem (**[Bck17]**) Let $p \in (0, 1]$ and $\alpha := 1/p - 1$. A function g is a **pointwise multiplier** of $\mathfrak{C}_\alpha(\mathbb{R}^n)$ **iff** $g \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$.

Proof: need to show that $g \in L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ **iff** for any $f \in \mathfrak{C}_\alpha(\mathbb{R}^n)$, it holds true that $fg \in \mathfrak{C}_\alpha(\mathbb{R}^n)$.

- **Sufficiency:**

$$\left|f(x)g(x) - P_{B,s}f(x)P_{B,s}g(x)\right|$$
$$\leq \left|f(x) - P_{B,s}f(x)\right| \, |g(x)| + \left|P_{B,s}f(x)\right| \left|g(x) - P_{B,s}g(x)\right| .$$

- **Necessity:** find some **subtle examples** of functions in $\mathfrak{C}_\alpha(\mathbb{R}^n)$. (Difficult)

$\square$

Facts on Pointwise Multipliers of $\mathfrak{C}_\alpha(\mathbb{R}^n)$/ §II

- $L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$ characterizes the **class of all pointwise multipliers of** $\mathfrak{C}_{1/p-1}(\mathbb{R}^n)$.
- $(L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n))^* = L^\infty(\mathbb{R}^n) \cap (H^{\Phi_p}(\mathbb{R}^n))^* = L^\infty(\mathbb{R}^n) \cap \mathfrak{C}_{\Phi_p}(\mathbb{R}^n)$.
- This **predicts** the following **sharp bilinear decomposition**: for any $p \in (0, 1)$,

$$H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n).$$

Wavelet Representation of Functions / §II

Let $E := \{0, 1\}^n \setminus \{(\overbrace{0, \ldots, 0}^{n \text{ times}})\}$ and $\mathcal{D}$ be the set of all dyadic cubes. For any $I \in \mathcal{D}$ and $\lambda \in E$, let ϕ_I and ψ_I^λ be, respectively, the **father** and the **mother wavelets** satisfying

- **Support condition:** $\operatorname{supp} \phi_I \subset mI$, $\operatorname{supp} \psi_I^\lambda \subset mI$ with m being a positive constant.
- **Cancelation condition:** $\int_{\mathbb{R}^n} \phi_I(x)\, dx = (2\pi)^{-1/2}$ and there exists $s \in \mathbb{Z}_+$ such that, for any multi-index α satisfying $|\alpha| \le s$, $\int_{\mathbb{R}^n} x^\alpha \psi_I^\lambda(x)\, dx = 0$.

Then, for any $f \in L^2(\mathbb{R}^n)$,

$$f = \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \langle f, \psi_I^\lambda \rangle \psi_I^\lambda.$$

Dobyinsky's Renormalizatoin (1) / §II

Let $f \times g \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then
$f \times g = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g) + \Pi_4(f, g)$, where

- $$\Pi_1(f, g) := \sum_{\substack{I, I' \in \mathcal{D} \\ |I| = |I'|}} \sum_{\lambda \in E} \langle f, \phi_I \rangle \langle g, \psi_{I'}^{\lambda} \rangle \phi_I \psi_{I'}^{\lambda},$$

- $$\Pi_2(f, g) := \sum_{\substack{I, I' \in \mathcal{D} \\ |I| = |I'|}} \sum_{\lambda \in E} \langle f, \psi_I^{\lambda} \rangle \langle g, \phi_{I'} \rangle \psi_I^{\lambda} \phi_{I'},$$

- $$\Pi_3(f, g) := \sum_{\substack{I, I' \in \mathcal{D} \\ |I| = |I'|}} \sum_{\substack{\lambda, \lambda' \in E \\ (I, \lambda) \neq (I', \lambda')}} \langle f, \psi_I^{\lambda} \rangle \langle g, \psi_{I'}^{\lambda'} \rangle \psi_I^{\lambda} \psi_{I'}^{\lambda'},$$

- $$\Pi_4(f, g) := \sum_{I \in \mathcal{D}} \sum_{\lambda \in E} \langle f, \psi_I^{\lambda} \rangle \langle g, \psi_I^{\lambda} \rangle \left(\psi_I^{\lambda}\right)^2.$$

Dobyinsky's Renormalizatoin (2) / §II

S. Dobyinsky, La “version ondelettes" du théoréme du Jacobien, **Rev. Mat. Iberoam. 11 (1995), 309-333**.

Boundedness of Operators $\{\Pi_i\}_{i=1}^4$ / §II

$$\boxed{f \times g = \Pi_1(f,\, g) + \Pi_2(f,\, g) + \Pi_3(f,\, g) + \Pi_4(f,\, g)}.$$

Let $p \in (0,\, 1)$ and $\alpha := 1/p - 1$.

- Π_1 and Π_3 can be extended to bilinear operators bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$.
- Π_4 can be extended to a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.
  - **Atomic decomposition** of $H^p(\mathbb{R}^n)$;
  - Simple **Hardy and Lebesgue estimates** for Π_i;
  - **Wavelet characterization** of $H^p(\mathbb{R}^n)$ and $\mathfrak{C}_\alpha(\mathbb{R}^n)$;
- Π_2 can be extended to a bilinear operator bounded from $H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n)$ to $H^{\Phi_p}(\mathbb{R}^n)$.

Bilinear Decomposition / §II

Let $p \in (0, 1)$. For any $f \in H^p(\mathbb{R}^n)$ and $g \in \mathfrak{C}_{1/p-1}(\mathbb{R}^n)$, it holds true that $f \times g = S(f, g) + T(f, g)$ with

- $S(f, g) := \Pi_4(f, g) \in L^1(\mathbb{R}^n)$.
- $T(f, g) := \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g) \in H^{\Phi_p}(\mathbb{R}^n)$.

Theorem ([Bck17]) Let $p \in (0, 1]$. Then the space $H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n)$ has the following **sharp bilinear decomposition**:

$$H^p(\mathbb{R}^n) \times \mathfrak{C}_{1/p-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n).$$

An Observation / §II

- Indeed, $\boxed{\Pi_2 : \ H^p(\mathbb{R}^n) \times \mathfrak{C}_\alpha(\mathbb{R}^n) \to H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)}$ and $H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n)$, where $H^p_{W_p}(\mathbb{R}^n)$ is the **weighted Hardy space associated** with the **weight** W_p defined by setting, for any $x \in \mathbb{R}^n$,

$$W_p(x) := \begin{cases} \dfrac{1}{(1+|x|)^{n(1-p)}} & \text{if } n[1/p-1] \notin \mathbb{Z}_+, \\ \dfrac{1}{(1+|x|)^{n(1-p)}\,[\log(e+|x|)]^p} & \text{if } n[1/p-1] \in \mathbb{Z}_+. \end{cases}$$

What is the relationship between $H^p_{W_p}(\mathbb{R}^n)$ and $H^{\Phi_p}(\mathbb{R}^n)$?

Intrinsic Structure of $H^{\Phi_p}(\mathbb{R}^n)$ (1) / §II

Let $\alpha := \frac{1}{p} - 1$. **Musielak-Orlicz growth function**: for any $(x, t) \in \mathbb{R}^n \times [0, \infty)$,

$$\Phi_p(x,\, t) := \begin{cases} \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}} & \text{if } n\alpha \notin \mathbb{Z}_+, \\ \dfrac{t}{\log(e+t) + [t(1+|x|)^n]^{1-p}[\log(e+|x|)]^p} & \text{if } n\alpha \in \mathbb{Z}_+, \end{cases}$$

Weight function: for any $x \in \mathbb{R}^n$,

$$W_p(x) := \begin{cases} \dfrac{1}{(1+|x|)^{n(1-p)}} & \text{if } n\alpha \notin \mathbb{Z}_+, \\ \dfrac{1}{(1+|x|)^{n(1-p)}\,[\log(e+|x|)]^p} & \text{if } n\alpha \in \mathbb{Z}_+. \end{cases}$$

Intrinsic Structure of $H^{\Phi_p}(\mathbb{R}^n)$ (2) / §II

Orlicz function: $\phi_0(t) := \frac{t}{\log(e+t)}$, $\forall\, t \in [0, \infty)$.

- $(\Phi_p(x,\, t))^{-1} = (\phi_0(t))^{-1} + (t^p W_p(x))^{-1}$, $\forall\, x \in \mathbb{R}^n$, $\forall\, t \in (0, \infty)$.

Theorem ([clyy])

- for $p \in (0,\, 1]$, $H^{\Phi_p}(\mathbb{R}^n) = H^{\phi_0}(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$;
- for $p \in (0,\, 1)$, $H^{\Phi_p}(\mathbb{R}^n) = H^1(\mathbb{R}^n) + H^p_{W_p}(\mathbb{R}^n)$.
- $H^p_{W_p}(\mathbb{R}^n) \subsetneqq H^{\Phi_p}(\mathbb{R}^n)$ with $\left[H^p_{W_p}(\mathbb{R}^n)\right]^* = \left[H^{\Phi_p}(\mathbb{R}^n)\right]^*$ when $p \in (0,\, 1)$.

Intrinsic Structure of $H^{\Phi_p}(\mathbb{R}^n)$ (3) / §II

Lemma (**[clyy]**) Let B be a **ball** in $\mathbb{R}^n$. Then

- for $p = 1$, $\|\chi_B\|^{-1}_{L^{\Phi_1}(\mathbb{R}^n)} \sim \|\chi_B\|^{-1}_{L^{\phi_0}(\mathbb{R}^n)} + \|\chi_B\|^{-1}_{L^1_{W_1}(\mathbb{R}^n)}$;
- for $p \in (0,\, 1)$, $\|\chi_B\|^{-1}_{L^{\Phi_p}(\mathbb{R}^n)} \sim \|\chi_B\|^{-1}_{L^p_{W_p}(\mathbb{R}^n)}$.

[clyy] **J. Cao, L. Liu, D. Yang and W. Yuan**, Intrinsic structures of certain Musielak-Orlicz-Hardy spaces, **J. Geom. Anal.** (to appear).

§III. Bilinear Decompositions of Products of **Local** Hardy and Lipschitz or BMO Spaces

Local Hardy Spaces $h^p(\mathbb{R}^n)$ / §III

- The **local Hardy space** $h^p(\mathbb{R}^n)$ for $p \in (0, 1]$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that their **quasi-norms**

$$\|f\|_{h^p(\mathbb{R}^n)} := \|f^*_{\mathrm{loc}}\|_{L^p(\mathbb{R}^n)} := \left\| \sup_{t\in(0,1)} (\varphi_t * f) \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\varphi_t * f(x) := \langle f, \frac{1}{t^n}\varphi(\frac{x-\cdot}{t})\rangle$ with $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Local BMO Spaces / §II

- The **local BMO space** $\mathrm{bmo}(\mathbb{R}^n)$ is defined via the **norm**

$$\|g\|_{\mathrm{bmo}(\mathbb{R}^n)} := \sup_{|B|<1} \left\{ \frac{1}{|B|} \int_B |g(x) - g_B| \, dx \right\} + \sup_{|B|\geq 1} \left\{ \frac{1}{|B|} \int_B |f(x)| \, dx \right\},$$

where $g_B := \frac{1}{|B|} \int_B g$.

Local Hardy Spaces and Their Dual Spaces / §III

- The **inhomogeneous Lipschitz space** $\Lambda_\alpha(\mathbb{R}^n)$ with $\alpha \in (0, 1)$ is defined via the **norm**

$$\|g\|_{\Lambda_\alpha(\mathbb{R}^n)} := \sup_{\substack{x,\, y\in\mathbb{R}^n \\ x\neq y}} \frac{|g(x) - g(y)|}{|x-y|^\alpha} + \|g\|_{L^\infty(\mathbb{R}^n)}.$$

- It is well known that

$$[h^p(\mathbb{R}^n)]^* = \begin{cases} \mathrm{bmo}(\mathbb{R}^n) & \text{when } p = 1, \\ \Lambda_\alpha(\mathbb{R}^n) & \text{when } p \in (\frac{n}{n+1}, 1) \end{cases}$$

with $\alpha := n(\frac{1}{p} - 1)$.

An variant of local Orlicz space/ §III

Let $\mathbb{Q}$ be an cube with side length 1. For any measurable function g on $\mathbb{Q}$, define the **Orlicz space** $L^{\phi_0}(\mathbb{Q})$ on $\mathbb{Q}$ by

$$\|g\|_{L^{\phi_0}(\mathbb{Q})} := \inf\left\{\lambda \in (0,\ \infty) :\ \int_{\mathbb{Q}} \phi_0\left(\frac{|g(x)|}{\lambda}\right) dx \le 1\right\}$$

with

$$\phi_0(t) := \frac{t}{\log(e+t)},\ \forall\, t \in [0, \infty).$$

A generalized Hölder's inequality:

$$\|fg\|_{L^{\phi_0}(\mathbb{Q})} \lesssim \|f\|_{L^1(\mathbb{Q})}\|g\|_{\mathrm{bmo}(\mathbb{R}^n)}.$$

A Variant of Local Orlicz-Hardy Space / §III

- Let $\phi_0(t) := \frac{t}{\log(e+t)}$ for any $t \in [0, \infty)$. For any measurable function g, let

$$\|g\|_{L_*^{\phi_0}(\mathbb{R}^n)} := \sum_{j \in \mathbb{Z}^n} \|g\|_{L^{\phi_0}(\mathbb{Q}_j)}$$

with $j := (j_1, \dots, j_n)$, $\mathbb{Q}_j := [j_1, j_1 + 1) \times \cdots \times [j_n, j_n + 1)$.

- The **local Orlicz-Hardy space** $h_*^{\phi_0}(\mathbb{R}^n)$ is defined by setting

$$h_*^{\phi_0}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f\|_{h_*^{\phi_0}(\mathbb{R}^n)} := \|f_{\rm loc}^*\|_{L_*^{\phi_0}(\mathbb{R}^n)} < \infty\}.$$

Bilinear Decompositions (1) / §III

For any $f \in h^1(\mathbb{R}^n)$ and $g \in \mathrm{bmo}(\mathbb{R}^n)$, $f \times g = S(f, g) + T(f, g)$ with

- $S(f, g) := \Pi_4(f, g) \in L^1(\mathbb{R}^n)$.
- $T(f, g) := \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g) \in h_*^{\phi_0}(\mathbb{R}^n)$.

[cky17] **J. Cao, L. D. Ky & D. Yang**, Bilinear decompositions of products of local Hardy and Lipschitz or BMO spaces through wavelets, **Commun. Contemp. Math.** (to appear).

Theorem (**[cky17]**). $h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n)$ has the following **bilinear decomposition**:

$$h^1(\mathbb{R}^n) \times \mathrm{bmo}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + h_*^{\phi_0}(\mathbb{R}^n).$$

- The **sharpness** of this decomposition is still unknown.

Bilinear Decompositions (2) / §III

Theorem (**[cky17]**) For any $p \in (\frac{n}{n+1}, 1)$ and $\alpha = \frac{1}{p} - 1$, $h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n)$ has the following **sharp bilinear decomposition**:

$$h^p(\mathbb{R}^n) \times \Lambda_\alpha(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + h^p(\mathbb{R}^n).$$

Theorem (**[cky17]**) Let $\mathbf{F} \in h^1(\mathbb{R}^n; \mathbb{R}^n)$ with $\operatorname{curl} \mathbf{F} \equiv 0$ and $\mathbf{G} \in \mathrm{bmo}(\mathbb{R}^n; \mathbb{R}^n)$ with $\operatorname{div} \mathbf{G} \equiv 0$. Then $\mathbf{F} \cdot \mathbf{G} \in h_*^{\Phi}(\mathbb{R}^n)$ with

$$\|\mathbf{F} \cdot \mathbf{G}\|_{h_*^{\phi_0}(\mathbb{R}^n)} \lesssim \|\mathbf{F}\|_{h^1(\mathbb{R}^n)} \|\mathbf{G}\|_{\mathrm{bmo}(\mathbb{R}^n)}.$$

- The last theorem when $p \in (\frac{n}{n+1}, 1)$ is still unknown.

§IV. Further Remarks

Spaces of Homogeneous Type

- A **quasi-metric space** $(\mathcal{X}, d)$ equipped with a nonnegative measure μ is called a **space of homogeneous type** if μ satisfies the following **measure doubling condition**: $\exists$ $C_{(\mathcal{X})} \in [1, \infty)$ such that, for any ball $B(x, r) := \{y \in \mathcal{X} : \ d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \leq C_{(\mathcal{X})}\mu(B(x, r)).$$

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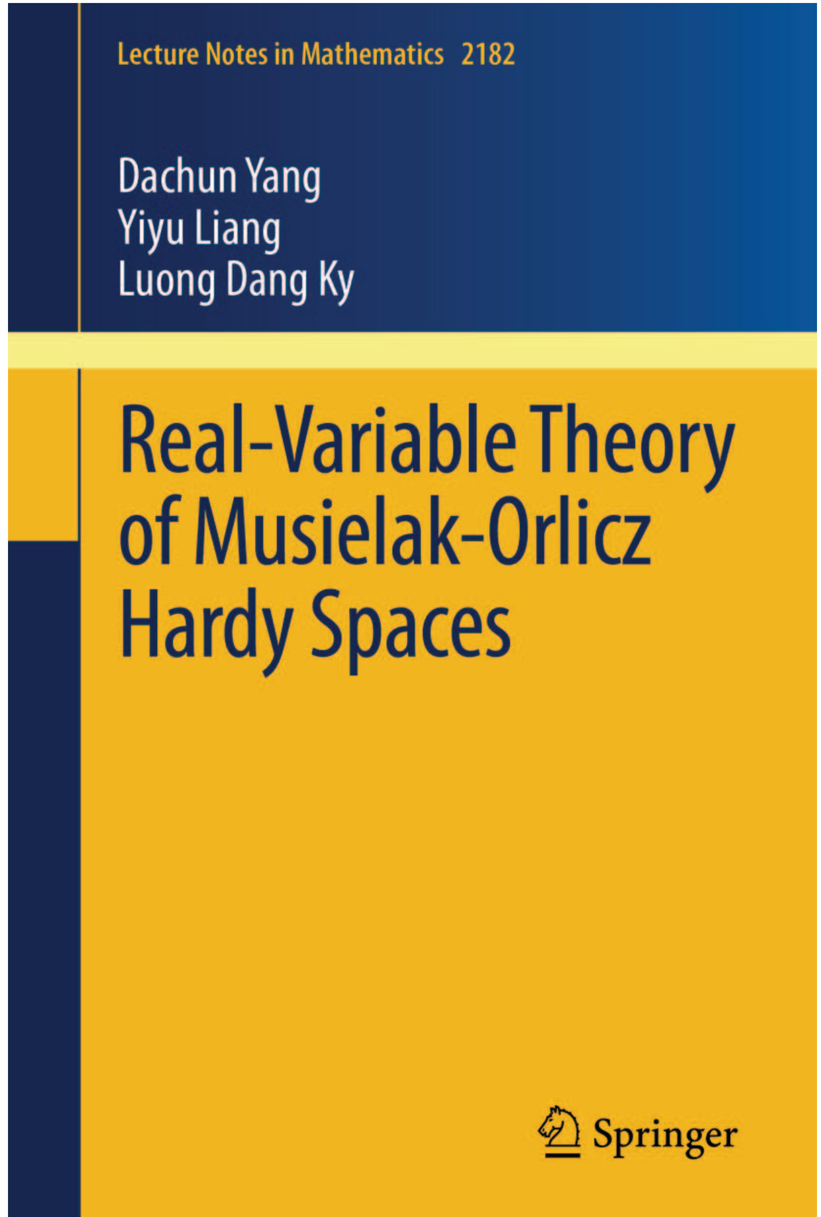

Thank you for your attention.