Some Monge-Ampère equations with degeneracy or singularities

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Conference on geometric analysis and nonlinear PDEs,
Harbin Institute of Technology, May 2019

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- Cheng-Yau: a proof arising from affine geometry;
- Caffarelli: viscosity solutions.

Some generalizations:

- Trudinger-Wang('00): the only convex open subset Ω of $\mathbb{R}^n$ which admits a convex C^2 solution of $\det \nabla^2 u = 1$ in Ω with

$$\lim_{x \to \partial\Omega} u(x) = \infty$$

is $\Omega = \mathbb{R}^n$;

Some generalizations:

- Caffarelli-Li ('03): if

$$\det \nabla^2 u = 1 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

then there exist $c \in \mathbb{R}, b \in \mathbb{R}^n, A \in \mathcal{M}_{n\times n}$ s.t.

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- Ferrer-Martínez-Milán ('00) for $n = 2$ (with an extra $\log \sqrt{x^T AX}$ term).

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- Caffarelli-Li ('04): if

$$\det \nabla^2 u = f \quad \text{in } \mathbb{R}^n$$

where f is periodic Hölder continuous, then there exist $b \in \mathbb{R}^n, A \in \mathcal{M}_{n\times n}$ s.t.

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- D. Li-Z.Li-Yuan ('17), D. Li-Z. Li('18), for special Lagrangian equations, half space, etc.

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0 is non-removable singular point of u_c if and only if $c > 0$.

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Theorem (J-Xiong '12)

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for some $c \geq 0$ *(modulo the unimodular affine equivalence).*

Next, local solutions to

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Describe the asymptotic behavior of u near the non-removable singularity $\{0\}$.

Theorem (J-Xiong '12)

Let $\Gamma \subset\subset \Omega$ *be either a point or a straight line segment. If a convex* $u \in C^2(\Omega \setminus \Gamma)$ *satisfies*

$$\det \nabla^2 u = 1 \text{ in } \Omega \setminus \Gamma,$$

then

$$|\nabla^2 u(x)| \leq \frac{C}{dist(x, \Gamma)}.$$

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- $\int_0^{|x|} (\tau^n + 1)^{\frac{1}{n}} \, \mathrm{d}\tau$ is of this rate.
- If $|\nabla^2 u(x)| = O(dist(x, \Gamma)^{-\alpha})$ for $\alpha \in (0, 1)$, then by Schulz-Wang, the singularity is removable.

Regularity:

Theorem

Let Ω *be a bounded convex domain,* $0 < \lambda \leq \Lambda < \infty$ *and* $\Gamma \subset\subset \Omega$*.*
Let $u \in C(\overline{\Omega})$ *be a generalized convex solution of*

$$\begin{aligned} \lambda \leq \det \nabla^2 u \leq \Lambda \qquad & \text{in } \Omega \setminus \Gamma, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

Then u *is locally strictly convex in* $\Omega \setminus \mathcal{C}(\Gamma)$*, where* $\mathcal{C}(\Gamma)$ *is the convex hull of* Γ*.*

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Question:

Is u strictly convex in $\Omega \setminus \{P_1, P_2\}$?

Existence and uniqueness:

Theorem (J-Xiong '12)

Let μ be a locally finite Borel measure s.t. the support of $(\mu - 1)$ is bounded. Then for every $c \in \mathbb{R}, b \in \mathbb{R}^n, A \in \mathcal{M}_{n\times n}$ s.t. $A > 0, \det A = 1$, there exists a unique convex solution of

$$\det \nabla^2 u = \mu \quad \textit{in } \mathbb{R}^n$$

$$\lim_{|x|\to+\infty} |u(x) - (\frac{1}{2}x^T Ax + b\cdot x + c)| = 0.$$

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Remark: If $\mathrm{d}\mu = f(x)\mathrm{d}x$ for some $f \in C(\mathbb{R}^n)$ satisfying $\mathrm{supp}(f - 1)$ is bounded and $\inf_{\mathbb{R}^n} f > 0$, then this was proved in Caffarelli-Li.

Brandolini, Nitsch, Salani and Trombetti extended Serrin's over derterminate result to $\sigma_k(\nabla^2 u)$: whenever Ω is a bounded smooth domain, and ν is the outer normal of $\partial\Omega$, if $u \in C^2(\overline{\Omega})$ is a solution of

$$\begin{cases} \sigma_k(\nabla^2 u) = \binom{n}{k} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial u/\partial\nu = 1 & \text{on } \partial\Omega \end{cases}$$

with $k = 1, 2, \cdots, n$, then after some translation Ω has to the unit ball and $u = \frac{|x|^2 - 1}{2}$.

We show that

Theorem (J-Xiong '12)

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$. If there exists a locally convex function $u \in C^1(\mathbb{R}^n \setminus \Omega) \cap C^2(\mathbb{R}^n \setminus \overline{\Omega})$ satisfying

$$\begin{cases} \det \nabla^2 u = 1 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u = 0 & \text{on } \partial\Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

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Remark: Not much is know for Serrin's problem in exterior domains (even assuming quadratic growth at infinity).

Degenerated Monge-Ampère equation

$$\det \nabla^2 u(x_1, x_2) = |x_1|^\alpha \quad \text{in } \mathbb{R}^2.$$

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appears, for instance, as a blowup limiting equation of

$$\det \nabla^2 u(x_1, x_2) = (x_1^2 + x_2^2)^{\alpha/2} \quad \text{in } B_1 \tag{1}$$

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They showed that the solution of (1) near 0 is either

- radial ($\sim |x|^{2+\frac{\alpha}{2}}$), or
- nonradial ($\sim c_1|x_1|^{2+\alpha} + c_2|x_2|^2 + h.o.t.$).

Theorem (J-Xiong '12)

Let u be a convex generalized solution of

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with $\alpha > -1$*. Then there exist some constants* $a > 0$*,* b *and a linear function* $\ell(x_1, x_2)$ *such that*

$$u(x_1, x_2) = \frac{a}{(\alpha+2)(\alpha+1)}|x_1|^{2+\alpha} + \frac{ab^2}{2}x_1^2 + bx_1x_2 + \frac{1}{2a}x_2^2 + \ell(x_1, x_2).$$

Regularity:

We needed to show that every solution of

$$\det \nabla^2 u(x_1, x_2) = |x_1|^\alpha \quad \text{in } \mathbb{R}^2$$

is strictly convex, so that $u \in C^{1,\delta}_{loc}(\mathbb{R}^2)$ and is smooth away from $\{x_1 = 0\}$.

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However, we have examples showing that it is not the case for local equations with $\alpha > 0$:

$$\det \nabla^2 u(x_1, x_2) = |x_1|^\alpha \quad \text{in } B_1.$$

(Write $u(x) = |x_1|^{\frac{2+\alpha}{2}} w(x_2)$ and solve for w).

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x_1, x_2) = (x_1, \nabla_{x_2} u(x)) =: (p_1, p_2).$$

T is injective. The partial Legendre transform $u^*(p)$ is

$$u^*(p) = x_2 \nabla_{x_2} u(x) - u(x).$$

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- u^* is concave w.r.t. p_1 and convex w.r.t. p_2;
- $(u^*)^* = u$;
- $u^*_{11} + |p_1|^\alpha u^*_{22} = 0$ in $T(\mathbb{R}^2)$.

Step 1: Prove

$$T(\mathbb{R}^2) = \mathbb{R}^2.$$

Hence

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Let $v = u^*_{22} \geq 0$. Then

$$v_{11} + |p_1|^\alpha v_{22} = 0 \quad \text{in } \mathbb{R}^2.$$

Step 2: The equation

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satisfies the Harnack inequality, and thus $v = u_{22}^*$ has to be a constant.

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$$u^*_{22} \equiv a, u^*_{11} \equiv -a|p_1|^\alpha, u^*_{112} = u^*_{122} = 0, u^*_{12} = b.$$

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Let

$$\phi(x_1, x_2) = |x_1|^{2+\alpha} + x_2^2 \quad \text{in } \mathbb{R}^2.$$

Then

$$(\nabla^2 \phi)^{1/2} = \begin{pmatrix} \sqrt{(2+\alpha)(1+\alpha)}|x_1|^{\alpha/2} & 0 \\ 0 & \sqrt{2} \end{pmatrix},$$

$$\det \nabla^2 \phi = 2(\alpha+2)(\alpha+1)|x_1|^{\alpha}.$$

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Note: $|x_1|^\alpha$ is A_∞ if $\alpha > -1$.

Let

$$A(x_1, x_2) = \begin{pmatrix} |x_1|^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$B := (\nabla^2 \phi)^{1/2} \cdot A \cdot (\nabla^2 \phi)^{1/2} = \begin{pmatrix} (2+\alpha)(1+\alpha) & 0 \\ 0 & 2 \end{pmatrix} > 0$$

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if $\alpha > -1$.Therefore, we can apply Caffarelli-Gutiérrez's Harnack inequality for linearized Monge-Ampère equations to

$$v_{11} + |p_1|^{\alpha} v_{22} = Tr(A\nabla^2 v) = 0, \quad \alpha > -1.$$

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We prove it by contradiction. Suppose that there exists $\bar{x}_1$ s.t.

$$\lim_{x_2 \to \infty} u_2(\bar{x}_1, x_2) := \beta < \infty.$$

Then

$$\lim_{x_2 \to \infty} u_2(x_1, x_2) = \beta \text{ for every } x_1 \in \mathbb{R}.$$

Proof of $T(\mathbb{R}^2) = \mathbb{R}^2$ (recall $T(x_1, x_2) = (x_1, \nabla_{x_2} u(x))$).

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We assume $\beta = 1$. Therefore,

$$T(\mathbb{R}^2) = (-\infty, \infty) \times (\beta_0, 1) \text{ for some } -\infty \leq \beta_0 < 1.$$

Recall

$$T(x_1, x_2) = (x_1, \nabla_{x_2} u(x)) =: (p_1, p_2).$$
$$T(\mathbb{R}^2) = (-\infty, \infty) \times (\beta_0, 1) \text{ for some } -\infty \leq \beta_0 < 1.$$

Since T is one-to-one and $u_2^*(p_1, p_2) = x_2$, we have

$$\lim_{p_2 \to 1^-} u_2^*(p_1, p_2) = \infty.$$

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$$\lim_{(p_1, p_2) \to (\bar{p}_1, 1)} u_2^*(p_1, p_2) = +\infty \ \forall \ \bar{p}_1 \in \mathbb{R}.$$

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Use comparison principle and the equation of u_2^* to show that this is impossible.

THANK YOU!