Finite Morse index solutions of Allen-Cahn equation

Kelei Wang

(based on joint work with Changfeng Gui and Juncheng Wei)

Wuhan University, China

wangkelei@whu.edu.cn

Harbin, 3 May 2019

The stability condition

Consider a functional

$$\mathcal{F}(u, \nabla u).$$

The stability condition

Consider a functional

$$\mathcal{F}(u, \nabla u).$$

We say u is a critical point if the first variation vanishes: given a family u^t for $t \in (-\varepsilon, \varepsilon)$ with $u^0 = u$,

$$\frac{d}{dt}\mathcal{F}(u^t, \nabla u^t)|_{t=0} = 0.$$

The stability condition

Consider a functional

$$\mathcal{F}(u, \nabla u).$$

We say u is a critical point if the first variation vanishes: given a family u^t for $t \in (-\varepsilon, \varepsilon)$ with $u^0 = u$,

$$\frac{d}{dt}\mathcal{F}(u^t, \nabla u^t)|_{t=0} = 0.$$

The second variation formula

$$\mathcal{Q}(\varphi) := \frac{d^2}{dt^2}\mathcal{F}(u^t, \nabla u^t)|_{t=0}, \quad \varphi := \frac{du^t}{dt}|_{t=0}.$$

Morse index = dim of the negative space for $\mathcal{Q}$. A solution is stable if its Morse index is 0.

Stable outside a compact set

If u is a critical point defined on the entire space $\mathbb{R}^n$, its Morse index is defined to be

$$\sup \dim\{X : \text{ finite dim subspace of } C_0^\infty(\mathbb{R}^n) \text{ with } \mathcal{Q}\lfloor_X < 0\}.$$

Stable outside a compact set

If u is a critical point defined on the entire space $\mathbb{R}^n$, its Morse index is defined to be

$$\sup\dim\{X: \text{ finite dim subspace of } C_0^\infty(\mathbb{R}^n) \text{ with } \mathcal{Q}\lfloor_X < 0\}.$$

Finite Morse index $\Longrightarrow$ stable outside a compact set, i.e. there exists a compact set K such that, for any $\varphi \in C_0^\infty(\mathbb{R}^n \setminus K)$,

$$\mathcal{Q}(\varphi) \geq 0.$$

Three models

- Minimal hypersurfaces

$$\text{Stability:} \quad \int_\Sigma |\nabla\varphi|^2 - |A|^2\varphi^2 \geq 0.$$

Three models

- Minimal hypersurfaces

$$\text{Stability:} \quad \int_\Sigma |\nabla\varphi|^2 - |A|^2\varphi^2 \geq 0.$$

- Allen-Cahn equation: $-\Delta u = u - u^3$

$$\text{Stability:} \quad \int_{\mathbb{R}^n} |\nabla\varphi|^2 + \left(3u^2 - 1\right)\varphi^2 \geq 0.$$

Three models

- Minimal hypersurfaces

$$\text{Stability:} \quad \int_\Sigma |\nabla\varphi|^2 - |A|^2\varphi^2 \geq 0.$$

- Allen-Cahn equation: $-\Delta u = u - u^3$

$$\text{Stability:} \quad \int_{\mathbb{R}^n} |\nabla\varphi|^2 + \left(3u^2 - 1\right)\varphi^2 \geq 0.$$

- Toda system: $\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}$

$$\text{Stability:} \quad \sum_k \int_{\mathbb{R}^n} |\nabla\varphi_k|^2 - \sqrt{2}e^{-\sqrt{2}(f_k - f_{k-1})}\left(\varphi_k - \varphi_{k-1}\right)^2 \geq 0.$$

Minimal hypersurfaces with finite Morse index

- Minimal surfaces in $\mathbb{R}^3$ with finite Morse index have finitely many ends (Fishcer-Colbrie, Gulliver in 1980s).

Minimal hypersurfaces with finite Morse index

- Minimal surfaces in $\mathbb{R}^3$ with finite Morse index have finitely many ends (Fishcer-Colbrie, Gulliver in 1980s).
- Minimal hypersurfaces with finite Morse index in high dimensional Euclidean spaces also have finitely many ends, but only in the topological sense. (H.-D. Cao-Y. Shen-S. Zhu '97, P. Li-J. Wang '02).

Bernstein problem

- **Graph version:** If $n \leq 7$, any entire solution to the minimal surface equation must be linear:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0, \quad \text{in } \mathbb{R}^n$$

Bernstein problem

- **Graph version:** If $n \leq 7$, any entire solution to the minimal surface equation must be linear:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad \text{in } \mathbb{R}^n$$

- **Minimizer version:** If $n \leq 7$, any hypersurface with minimizing area in $\mathbb{R}^n$ must be a hyperplane.

$$\text{Area}\left(\Sigma \cap B_R\right) \leq \text{Area}\left(\widetilde{\Sigma} \cap B_R\right), \quad \text{if } \Sigma = \widetilde{\Sigma} \text{ outside} B_R.$$

Bernstein problem

- **Graph version:** If $n \leq 7$, any entire solution to the minimal surface equation must be linear:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0, \quad \text{in } \mathbb{R}^n$$

- **Minimizer version:** If $n \leq 7$, any hypersurface with minimizing area in $\mathbb{R}^n$ must be a hyperplane.

$$\text{Area}\left(\Sigma \cap B_R\right) \leq \text{Area}\left(\widetilde{\Sigma} \cap B_R\right), \quad \text{if } \ \Sigma = \widetilde{\Sigma} \text{ outside} B_R.$$

- **Stable version:** If $n \leq 7$, any stable minimal hypersurface in $\mathbb{R}^n$ must be a hyperplane. ($n = 3$ by do Carmo and Peng, Fischer-Colbrie and Schoen, Pogorolev around 1980.)

De Giorgi conjecture

- **Graph version:** Let $u \in C^2(\mathbb{R}^n)$ be a solution of the Allen-Cahn equation, such that $\frac{\partial u}{\partial x_n} > 0$ in $\mathbb{R}^n$. Is it true that u is one dimensional if $n \leq 8$?
  Ghoussoub-Gui, Ambrosio-Cabre, Savin, Del Pino-Kowalczyk-Wei etc.

De Giorgi conjecture

- **Graph version:** Let $u \in C^2(\mathbb{R}^n)$ be a solution of the Allen-Cahn equation, such that $\frac{\partial u}{\partial x_n} > 0$ in $\mathbb{R}^n$. Is it true that u is one dimensional if $n \leq 8$?
  Ghoussoub-Gui, Ambrosio-Cabre, Savin, Del Pino-Kowalczyk-Wei etc.
- **Minimizer version:** A minimizing solution of (AC) in $\mathbb{R}^n$ is one dimensional if $n \leq 7$ (Savin '09).

De Giorgi conjecture

- **Graph version:** Let $u \in C^2(\mathbb{R}^n)$ be a solution of the Allen-Cahn equation, such that $\frac{\partial u}{\partial x_n} > 0$ in $\mathbb{R}^n$. Is it true that u is one dimensional if $n \leq 8$?

  Ghoussoub-Gui, Ambrosio-Cabre, Savin, Del Pino-Kowalczyk-Wei etc.
- **Minimizer version:** A minimizing solution of (AC) in $\mathbb{R}^n$ is one dimensional if $n \leq 7$ (Savin '09).
- **Stable version:** A stable solution of (AC) in $\mathbb{R}^n$ is $1D$ if $n \leq 7$.

De Giorgi conjecture

- **Graph version:** Let $u \in C^2(\mathbb{R}^n)$ be a solution of the Allen-Cahn equation, such that $\frac{\partial u}{\partial x_n} > 0$ in $\mathbb{R}^n$. Is it true that u is one dimensional if $n \leq 8$?

  Ghoussoub-Gui, Ambrosio-Cabre, Savin, Del Pino-Kowalczyk-Wei etc.
- **Minimizer version:** A minimizing solution of (AC) in $\mathbb{R}^n$ is one dimensional if $n \leq 7$ (Savin '09).
- **Stable version:** A stable solution of (AC) in $\mathbb{R}^n$ is $1D$ if $n \leq 7$.

Stable De Giorgi is only known for $n = 2$: quadratic area growth or parabolicity.

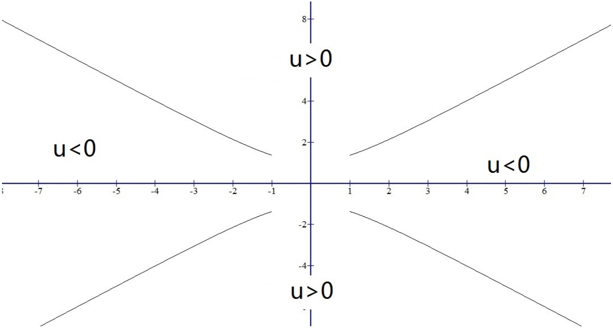
The structure of finite Morse index solutions

Theorem (W.-Wei '17)

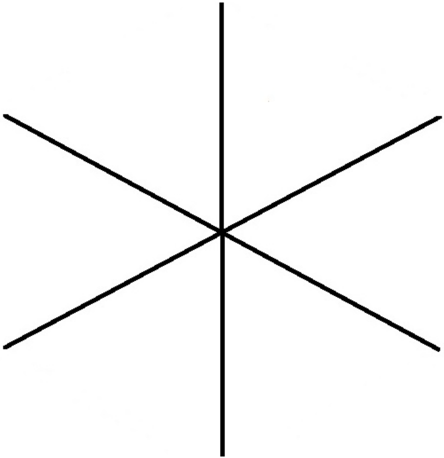
A finite Morse index solution of the Allen-Cahn equation in $\mathbb{R}^2$ *has finitely many ends.*

Solutions with finitely many ends

u looks like the $1D$ solution along each end. $\Longleftarrow$ Refined asymptotics, exponential convergence (Gui '08, Del Pino-Kowalczyk-Pacard '13).



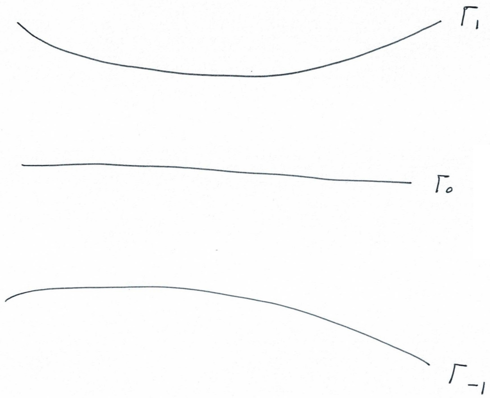
Example: Pizza solutions



Example: Toda solutions

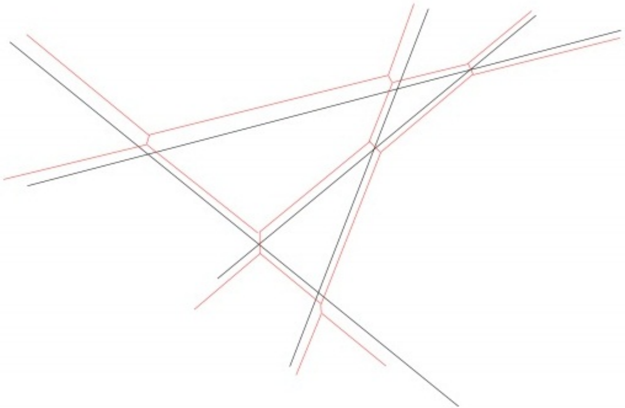
Del Pino-Kowalczyk-Pacard-Wei '10: $\{u = 0\}$ is close to the graph of Toda solutions:

$$f_k''(x) = e^{-\sqrt{2}[f_k(x) - f_{k-1}(x)]} - e^{-\sqrt{2}[f_{k+1}(x) - f_k(x)]}, \quad x \in \mathbb{R}, \quad 1 \le k \le Q.$$



Example: solutions with many ends

Kowalczyk-Y. Liu-Pacard-Wei '15:



A finiteness result for nodal domains

Let u_e be the directional derivative in e-direction, which satisfies the linearized equation

$$\Delta u_e = W''(u)u_e.$$

Lemma

If the Morse index is N, the number of connected components of $\{u_e \neq 0\}$ is at most $2N$.

A finiteness result for nodal domains

Let u_e be the directional derivative in e-direction, which satisfies the linearized equation

$$\Delta u_e = W''(u)u_e.$$

Lemma

If the Morse index is N, the number of connected components of $\{u_e \neq 0\}$ is at most $2N$.

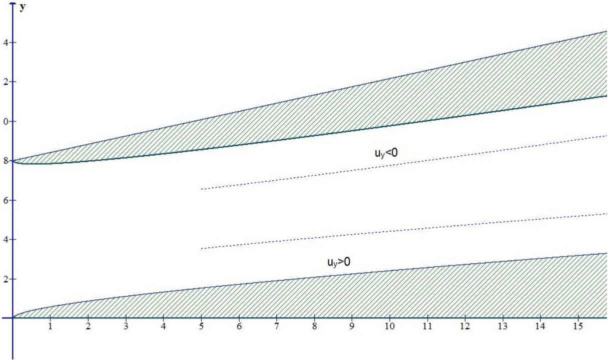
- Liouville theorem for the degenerate equation

$$\operatorname{div}\left(\varphi^2 \nabla \frac{u_e}{\varphi}\right) = 0.$$

- Similar to Courant's nodal domain theorem: entire space? $n = 2 \Longrightarrow$ log cut-off functions, Ambrosio-Cabré '03...

Transferring finiteness information

If each end of $\{u = 0\}$ has an asymptotic direction at infinity, finiteness of nodal domains of u_e can be transformed into finiteness of ends.



Curvature decay

Theorem (W.-Wei '17)

Let u be a finite Morse index solution of the Allen-Cahn equation in $\mathbb{R}^2$. For all x large,

$$|A(x)|^2 := \frac{|\nabla^2 u(x)|^2 - |\nabla|\nabla u(x)||^2}{|\nabla u(x)|^2} \leq \frac{C}{|x|^2}.$$

Curvature decay

Theorem (W.-Wei '17)

Let u be a finite Morse index solution of the Allen-Cahn equation in $\mathbb{R}^2$. For all x large,

$$|A(x)|^2 := \frac{|\nabla^2 u(x)|^2 - |\nabla|\nabla u(x)||^2}{|\nabla u(x)|^2} \le \frac{C}{|x|^2}.$$

Theorem (Schoen '83)

Given a three dimensional manifold M (with some curvature bounds). Let Σ be a stable immersed minimal surface in a ball $B_R(p) \subset M$ with $\partial\Sigma \subset \partial B_R(p)$. Then

$$\sup_{B_{R/2}(p)\cap\Sigma} |A_\Sigma|^2 \le \frac{C}{R^2}.$$

Sternberg-Zumbrun inequality

$$\text{Stability} \Leftrightarrow \int |\nabla\varphi|^2|\nabla u|^2 \geq \int \varphi^2 \left[|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right].$$

- $|\nabla u|^2 dx$ corresponds to the area measure of minimal surfaces.

Sternberg-Zumbrun inequality

$$\text{Stability} \Leftrightarrow \int |\nabla\varphi|^2|\nabla u|^2 \geq \int \varphi^2 \left[|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right].$$

- $|\nabla u|^2 dx$ corresponds to the area measure of minimal surfaces.
- If $|\nabla u| \neq 0$,

$$\frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2} = |A|^2 + |\nabla_T \log|\nabla u||^2,$$

where A is the second fundamental form of level sets $\{u = const.\}$ and ∇_T is the tangential derivatives along these level sets.

Sternberg-Zumbrun inequality

$$\text{Stability} \Leftrightarrow \int |\nabla\varphi|^2|\nabla u|^2 \geq \int \varphi^2 \left[|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right].$$

- $|\nabla u|^2 dx$ corresponds to the area measure of minimal surfaces.
- If $|\nabla u| \neq 0$,

$$\frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|^2} = |A|^2 + |\nabla_T \log |\nabla u||^2,$$

  where A is the second fundamental form of level sets $\{u = const.\}$ and ∇_T is the tangential derivatives along these level sets.
- Simons inequality for this curvature term? Not found yet. Seems to be a common difficulty in semilinear problems.

A blow up proof

- Stability outside $B_R(0) \Longleftarrow$ Finite Morse index.

A blow up proof

- Stability outside $B_R(0) \Longleftarrow$ Finite Morse index.
- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$.

A blow up proof

- Stability outside $B_R(0) \Longleftarrow$ Finite Morse index.
- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$.
- Find y_k satisfying

$$|A(y_k)| \geq |A(x_k)|, \qquad |A(y_k)||y_k| \geq k,$$

$$|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_{k|A(y_k)|^{-1}}(y_k).$$

A blow up proof

- Stability outside $B_R(0) \Longleftarrow$ Finite Morse index.
- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$.
- Find y_k satisfying

$$|A(y_k)| \geq |A(x_k)|, \qquad |A(y_k)||y_k| \geq k,$$

$$|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_{k|A(y_k)|^{-1}}(y_k).$$

- Let $\varepsilon_k := |A(y_k)|$ and define $u_k(x) := u(y_k + \varepsilon_k^{-1} x)$. $|y_k| \to +\infty$ and $\varepsilon_k \to 0 \Longleftarrow$ Locally close to $1D$ solution, by stable De Giorgi for $n = 2$.

A blow up proof

- Stability outside $B_R(0) \Longleftarrow$ Finite Morse index.
- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$.
- Find y_k satisfying

$$|A(y_k)| \geq |A(x_k)|, \qquad |A(y_k)||y_k| \geq k,$$

$$|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_{k|A(y_k)|^{-1}}(y_k).$$

- Let $\varepsilon_k := |A(y_k)|$ and define $u_k(x) := u(y_k + \varepsilon_k^{-1} x)$. $|y_k| \to +\infty$ and $\varepsilon_k \to 0 \Longleftarrow$ Locally close to $1D$ solution, by stable De Giorgi for $n = 2$.
- In $B_k(0)$, u_k is a stable solution of (AC) with parameter ε_k.

A blow up proof

- Stability outside $B_R(0) \Longleftarrow$ Finite Morse index.
- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$.
- Find y_k satisfying

$$|A(y_k)| \geq |A(x_k)|, \qquad |A(y_k)||y_k| \geq k,$$

$$|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_{k|A(y_k)|^{-1}}(y_k).$$

- Let $\varepsilon_k := |A(y_k)|$ and define $u_k(x) := u(y_k + \varepsilon_k^{-1}x)$. $|y_k| \to +\infty$ and $\varepsilon_k \to 0 \Longleftarrow$ Locally close to $1D$ solution, by stable De Giorgi for $n = 2$.
- In $B_k(0)$, u_k is a stable solution of (AC) with parameter ε_k.
- The curvature of $\{u_{\varepsilon_k} = 0\}$ is uniformly bounded, and it equals 1 at the origin.

Second order regularity

Theorem (W.-Wei '18)

Let u_ε be a sequence of stable solutions to (AC) such that $\{u_\varepsilon = 0\}$ are uniformly $C^{1,\beta}$ for some $\beta \in (0,1)$. If $n \leq 10$, then $\{u_\varepsilon = 0\}$ are uniformly bounded in $C^{2,\alpha}$ for any $\alpha \in (0,1)$. Moreover, the mean curvature is of the order $O(\varepsilon^\alpha)$.

Second order regularity

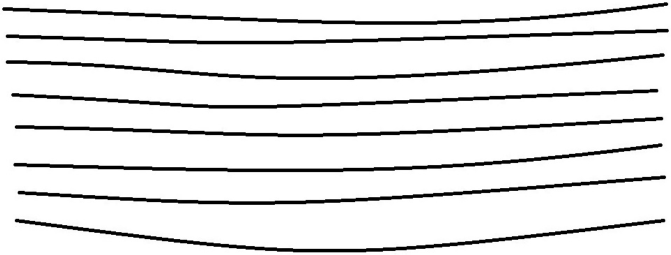
Theorem (W.-Wei '18)

Let u_ε be a sequence of stable solutions to (AC) such that $\{u_\varepsilon = 0\}$ are uniformly $C^{1,\beta}$ for some $\beta \in (0,1)$. If $n \leq 10$, then $\{u_\varepsilon = 0\}$ are uniformly bounded in $C^{2,\alpha}$ for any $\alpha \in (0,1)$. Moreover, the mean curvature is of the order $O(\varepsilon^\alpha)$.

Chodosh-Mantoulidis '18 has obtained the same result in dimension 3, which was used in their study of min-max minimal surfaces in three manifolds (Multiplicity one conjecture of Marques-Neves, existence of infinitely many minimal surfaces in generic case).

Clustering interfaces

There could be more and more connected components of $\{u_\varepsilon = 0\}$, which can collapse to the same limit. $\Longrightarrow$ clustering interfaces.



Toda system

Curvature bound on $\{u_\varepsilon = 0\} \Longrightarrow \{u_\varepsilon = 0\}$ locally represented by graphs $\cup_k \Gamma_{k,\varepsilon}$, where

$$\Gamma_{k,\varepsilon} = \{x_2 = f_{k,\varepsilon}(x_1)\}, \quad \cdots < f_{k-1,\varepsilon} < f_{k,\varepsilon} < f_{k+1,\varepsilon} < \cdots .$$

The cardinality of index set could go to infinity.

Toda system

Curvature bound on $\{u_\varepsilon = 0\} \Longrightarrow \{u_\varepsilon = 0\}$ locally represented by graphs $\cup_k \Gamma_{k,\varepsilon}$, where

$$\Gamma_{k,\varepsilon} = \{x_2 = f_{k,\varepsilon}(x_1)\}, \quad \cdots < f_{k-1,\varepsilon} < f_{k,\varepsilon} < f_{k+1,\varepsilon} < \cdots .$$

The cardinality of index set could go to infinity.

Interaction between different interfaces has the form

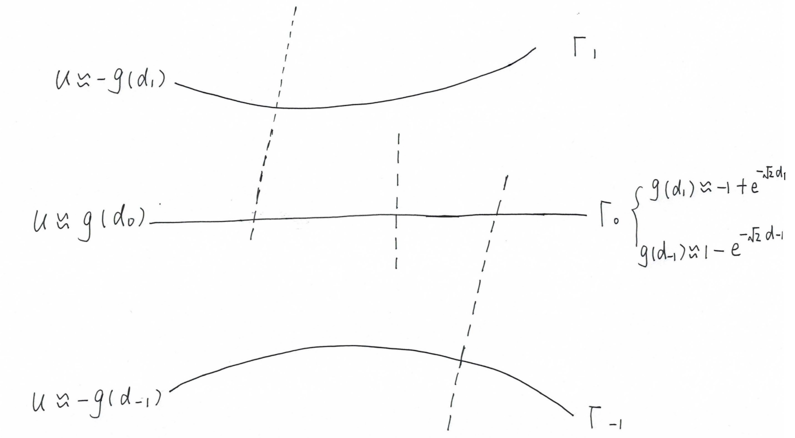
$$\operatorname{div}\left(\frac{\nabla f_{k,\varepsilon}}{\sqrt{1+|\nabla f_{k,\varepsilon}|^2}}\right) = \frac{A}{\varepsilon}\left[e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon}-f_{k-1,\varepsilon}\right)} - e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k+1,\varepsilon}-f_{k,\varepsilon}\right)}\right] + h.o.t.$$

Infinite dimensional Lyapunov-Schmidt reduction of Del Pino, Kowalczyk, Wei and their collaborators.

Interaction between transition layers

Approximate solution: near Γ_k,

$$g_* := g_k + \sum_{\ell<k}\left[g_\ell - (-1)^\ell\right] + \sum_{\ell>k}\left[g_\ell + (-1)^\ell\right].$$



Obstruction to $C^{2,\alpha}$ estimates of $f_{k,\varepsilon}$

$$\Delta f_{k,\varepsilon} = \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k+1,\varepsilon} - f_{k,\varepsilon}\right)}.$$

$$f_{k+1,\varepsilon} - f_{k,\varepsilon} \geq \frac{\sqrt{2}\,(1+\alpha)}{2}\varepsilon|\log\varepsilon| - C\varepsilon \Longrightarrow f_{k,\varepsilon} \in C^{2,\alpha}.$$

Obstruction to $C^{2,\alpha}$ estimates of $f_{k,\varepsilon}$

$$\Delta f_{k,\varepsilon} = \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k+1,\varepsilon} - f_{k,\varepsilon}\right)}.$$

$$f_{k+1,\varepsilon} - f_{k,\varepsilon} \geq \frac{\sqrt{2}\,(1+\alpha)}{2}\varepsilon|\log\varepsilon| - C\varepsilon \Longrightarrow f_{k,\varepsilon} \in C^{2,\alpha}.$$

On the other hand, if

$$f_{k+1,\varepsilon} - f_{k,\varepsilon} \leq \frac{\sqrt{2}}{2}\varepsilon|\log\varepsilon| + C\varepsilon,$$

define the blow up sequence

$$\widetilde{f}_{k,\varepsilon}(x) := \frac{1}{\varepsilon} f_{k,\varepsilon}\left(\varepsilon^{\frac{1}{2}} x\right) - \frac{\sqrt{2}\alpha}{2}|\log\varepsilon|.$$

They converge to an entire solution of the Toda system

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}, \quad \text{in } \mathbb{R}^{n-1}.$$

Reduction of the stability condition

- If u_ε is stable, $(f_{k,\varepsilon})$ satisfies a stability condition:

$$\sum_k \int |\nabla \eta_k|^2 \geq \frac{\sqrt{2}A}{\varepsilon^2} \sum_k \int (\eta_k - \eta_{k-1})^2 \, e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - h.o.t..$$

Reduction of the stability condition

- If u_ε is stable, $(f_{k,\varepsilon})$ satisfies a stability condition:

$$\sum_k \int |\nabla \eta_k|^2 \geq \frac{\sqrt{2}A}{\varepsilon^2} \sum_k \int (\eta_k - \eta_{k-1})^2 e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - h.o.t..$$

- Uniform $C^{2,\alpha}$ estimates of clustering interfaces for stable solutions of (AC)

Reduction of the stability condition

- If u_ε is stable, $(f_{k,\varepsilon})$ satisfies a stability condition:

$$\sum_k \int |\nabla \eta_k|^2 \geq \frac{\sqrt{2}A}{\varepsilon^2} \sum_k \int (\eta_k - \eta_{k-1})^2 \, e^{-\frac{\sqrt{2}}{\varepsilon}\left(f_{k,\varepsilon} - f_{k-1,\varepsilon}\right)} - h.o.t..$$

- Uniform $C^{2,\alpha}$ estimates of clustering interfaces for stable solutions of (AC)
  $\Longleftarrow$ Non-existence of entire stable solutions of Toda system.

Liouville theorems for Toda system

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}$$

Toda system if $k = 1, \cdots, Q$, Toda lattice if k runs over $\mathbb{Z}$.

Theorem (W. '19)

- *If $n < 10$, there does not exist entire stable solution of Toda lattice in $\mathbb{R}^n$.*

Liouville theorems for Toda system

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}$$

Toda system if $k = 1, \cdots, Q$, Toda lattice if k runs over $\mathbb{Z}$.

Theorem (W. '19)

- *If $n < 10$, there does not exist entire stable solution of Toda lattice in $\mathbb{R}^n$.*
- *If $1 \le n < 4Q(Q-1) + 2$, there is no entire stable solution of Q-component Toda system in $\mathbb{R}^n$.*

Liouville theorems for Toda system

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}$$

Toda system if $k = 1, \cdots, Q$, Toda lattice if k runs over $\mathbb{Z}$.

Theorem (W. '19)

- *If $n < 10$, there does not exist entire stable solution of Toda lattice in $\mathbb{R}^n$.*
- *If $1 \leq n < 4Q(Q-1) + 2$, there is no entire stable solution of Q-component Toda system in $\mathbb{R}^n$.*
- *If $3 \leq n \leq 9$, there is no solution of Toda lattice in $\mathbb{R}^n$, which is stable outside a compact set.*

Liouville theorems for Toda system

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)}$$

Toda system if $k = 1, \cdots, Q$, Toda lattice if k runs over $\mathbb{Z}$.

Theorem (W. '19)

- *If $n < 10$, there does not exist entire stable solution of Toda lattice in $\mathbb{R}^n$.*
- *If $1 \leq n < 4Q(Q-1) + 2$, there is no entire stable solution of Q-component Toda system in $\mathbb{R}^n$.*
- *If $3 \leq n \leq 9$, there is no solution of Toda lattice in $\mathbb{R}^n$, which is stable outside a compact set.*
- *If $3 \leq n < N_Q$, there is no finite Morse index solution of Q-component Toda system in $\mathbb{R}^n$.*

An ε-regularity theorem

Theorem (W. '19)

For any n, there exists a universal constant η such that, if (f_k) is a stable solution to the Toda lattice

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)} \quad \text{in } B_1 \subset \mathbb{R}^n,$$

then

$$\int_{B_1} e^{-\sqrt{2}(f_k - f_{k-1})} \leq \eta(n) \quad \Longrightarrow \quad \sup_{B_{1/2}} e^{-\sqrt{2}(f_k - f_{k-1})} \leq \frac{1}{2}.$$

An ε-regularity theorem

Theorem (W. '19)

For any n, there exists a universal constant η such that, if (f_k) is a stable solution to the Toda lattice

$$\Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)} \quad \text{in } B_1 \subset \mathbb{R}^n,$$

then

$$\int_{B_1} e^{-\sqrt{2}(f_k - f_{k-1})} \leq \eta(n) \quad \Longrightarrow \quad \sup_{B_{1/2}} e^{-\sqrt{2}(f_k - f_{k-1})} \leq \frac{1}{2}.$$

Applying this ε-regularity to suitable rescalings of Toda system constructed from (AC), gives a decay estimate on $e^{-\frac{\sqrt{2}}{\varepsilon}(f_{k,\varepsilon} - f_{k-1,\varepsilon})}$ in shrinking balls, leading finally to

$$e^{-\frac{\sqrt{2}}{\varepsilon}(f_{k,\varepsilon} - f_{k-1,\varepsilon})} \lesssim \varepsilon^{1+\alpha}, \qquad \text{in the interior.}$$

Prototype results

All results on Toda system are based on corresponding results on the Liouville equation

$$-\Delta u = e^u, \quad \text{in } \mathbb{R}^n.$$

- Farina '07: There does not exist stable entire solution if $n \leq 9$;
- Dancer-Farina '09: There does not exist finite Morse index entire solution if $3 \leq n \leq 9$;
- W. '12: ε-regularity for stable solutions of Liouville equation.

Axially symmetric solutions

Denote points in $\mathbb{R}^{n+1}$ by $(x_1, \cdots, x_n, z)$ and let $r := \sqrt{x_1^2 + \cdots + x_n^2}$. A function u is axially symmetric if $u(x_1, \cdots, x_n, z) = u(r, z)$.

Theorem (Gui-Wei-W. '19)

If $3 \le n \le 9$, any axially symmetric solution of (AC) in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $\mathcal{C}_R$, depends only on z, i.e. it is a $1D$ solution in the z-direction.

The critical $3D$ case

Theorem (Gui-Wei-W. '19)

An axially symmetric solution of (AC) in $\mathbb{R}^3$ with finite Morse index has finitely many ends and quadratic energy growth

$$\int_{B_R(x)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq CR^2. \tag{1}$$

The critical $3D$ case

Theorem (Gui-Wei-W. '19)

An axially symmetric solution of (AC) in $\mathbb{R}^3$ with finite Morse index has finitely many ends and quadratic energy growth

$$\int_{B_R(x)}\left[\frac{1}{2}|\nabla u|^2+W(u)\right]\leq CR^2. \tag{1}$$

Theorem (Gui-Wei-W. '19)

Suppose u is an axially symmetric solution of (AC) in $\mathbb{R}^3$.

- *If it is stable, it depends only on z $\Longleftrightarrow$ the solution has one end.*

The critical $3D$ case

Theorem (Gui-Wei-W. '19)

An axially symmetric solution of (AC) in $\mathbb{R}^3$ with finite Morse index has finitely many ends and quadratic energy growth

$$\int_{B_R(x)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq CR^2. \tag{1}$$

Theorem (Gui-Wei-W. '19)

Suppose u is an axially symmetric solution of (AC) in $\mathbb{R}^3$.

- *If it is stable, it depends only on $z \iff$ the solution has one end.*
- *If its Morse index equals 1, it has exactly two ends.*

The critical $3D$ case

Theorem (Gui-Wei-W. '19)

An axially symmetric solution of (AC) in $\mathbb{R}^3$ with finite Morse index has finitely many ends and quadratic energy growth

$$\int_{B_R(x)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq CR^2. \tag{1}$$

Theorem (Gui-Wei-W. '19)

Suppose u is an axially symmetric solution of (AC) in $\mathbb{R}^3$.

- *If it is stable, it depends only on $z \iff$ the solution has one end.*
- *If its Morse index equals 1, it has exactly two ends.*

The structure and moduli space of two-end solutions in $\mathbb{R}^3$ have been studied in detail by Gui-Liu-Wei '17 $\Longrightarrow$ Catenoid or Liouville equation.

Thanks for your attention!