Finite Morse index solutions of Allen-Cahn equation

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Consider a functional

\[ \mathcal{F}(u, \nabla u). \]
The stability condition

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We say \( u \) is a critical point if the first variation vanishes: given a family \( u^t \) for \( t \in (-\varepsilon, \varepsilon) \) with \( u^0 = u \),
\[ \frac{d}{dt} \mathcal{F}(u^t, \nabla u^t)|_{t=0} = 0. \]
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\[ \frac{d}{dt} F(u^t, \nabla u^t)|_{t=0} = 0. \]

The second variation formula

\[ Q(\varphi) := \frac{d^2}{dt^2} F(u^t, \nabla u^t)|_{t=0}, \quad \varphi := \frac{du^t}{dt}|_{t=0}. \]

Morse index = dim of the negative space for \( Q \). A solution is stable if its Morse index is 0.
If $u$ is a critical point defined on the entire space $\mathbb{R}^n$, its Morse index is defined to be

$$\sup \dim \{X : \text{finite dim subspace of } C_0^\infty(\mathbb{R}^n) \text{ with } Q|_X < 0\}.$$
Stable outside a compact set

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Finite Morse index $\iff$ stable outside a compact set, i.e. there exists a compact set $K$ such that, for any $\varphi \in C_0^\infty(\mathbb{R}^n \setminus K)$,

$$Q(\varphi) \geq 0.$$
Three models

- Minimal hypersurfaces

Stability:

\[
\int_{\Sigma} |\nabla \varphi|^2 - |A|^2 \varphi^2 \geq 0.
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- Allen-Cahn equation: \(-\Delta u = u - u^3\)

  Stability: \[ \int_{\mathbb{R}^n} |\nabla \varphi|^2 + (3u^2 - 1) \varphi^2 \geq 0. \]
Three models

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- **Toda system:** \( \Delta f_k = e^{-\sqrt{2}(f_k - f_{k-1})} - e^{-\sqrt{2}(f_{k+1} - f_k)} \)

  Stability: \[ \sum_{k} \int_{\mathbb{R}^n} |\nabla \varphi_k|^2 - \sqrt{2}e^{-\sqrt{2}(f_k - f_{k-1})} (\varphi_k - \varphi_{k-1})^2 \geq 0. \]
Minimal surfaces in $\mathbb{R}^3$ with finite Morse index have finitely many ends (Fishcer-Colbrie, Gulliver in 1980s).
Minimal hypersurfaces with finite Morse index

- Minimal surfaces in $\mathbb{R}^3$ with finite Morse index have finitely many ends (Fishcer-Colbrie, Gulliver in 1980s).
- Minimal hypersurfaces with finite Morse index in high dimensional Euclidean spaces also have finitely many ends, but only in the topological sense. (H.-D. Cao-Y. Shen-S. Zhu ’97, P. Li-J. Wang ’02).
Bernstein problem

- **Graph version:** If $n \leq 7$, any entire solution to the minimal surface equation must be linear:

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0, \quad \text{in } \mathbb{R}^n$$
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- **Minimizer version**: If \( n \leq 7 \), any hypersurface with minimizing area in \( \mathbb{R}^n \) must be a hyperplane.

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\text{Area} \left( \Sigma \cap B_R \right) \leq \text{Area} \left( \tilde{\Sigma} \cap B_R \right), \quad \text{if } \Sigma = \tilde{\Sigma} \text{ outside } B_R.
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- **Stable version:** If \( n \leq 7 \), any stable minimal hypersurface in \( \mathbb{R}^n \) must be a hyperplane. (\( n = 3 \) by do Carmo and Peng, Fischer-Colbrie and Schoen, Pogorolev around 1980.)
De Giorgi conjecture

- **Graph version:** Let \( u \in C^2(\mathbb{R}^n) \) be a solution of the Allen-Cahn equation, such that \( \frac{\partial u}{\partial x_n} > 0 \) in \( \mathbb{R}^n \). Is it true that \( u \) is one dimensional if \( n \leq 8 \)?

  Ghoussoub-Gui, Ambrosio-Cabre, Savin, Del Pino-Kowalczyk-Wei etc.
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The structure of finite Morse index solutions

Theorem (W.-Wei ’17)

A finite Morse index solution of the Allen-Cahn equation in $\mathbb{R}^2$ has finitely many ends.
Solutions with finitely many ends

$u$ looks like the 1D solution along each end. $\Leftarrow$ Refined asymptotics, exponential convergence (Gui ’08, Del Pino-Kowalczyk-Pacard ’13).
Example: Pizza solutions
Example: Toda solutions

Del Pino-Kowalczyk-Pacard-Wei '10: \( \{ u = 0 \} \) is close to the graph of Toda solutions:

\[
f_k''(x) = e^{-\sqrt{2}[f_k(x)-f_{k-1}(x)]} - e^{-\sqrt{2}[f_{k+1}(x)-f_k(x)]}, \quad x \in \mathbb{R}, \quad 1 \leq k \leq Q.
\]
Example: solutions with many ends

Kowalczyk-Y. Liu-Pacard-Wei ’15:
A finiteness result for nodal domains

Let $u_e$ be the directional derivative in $e$-direction, which satisfies the linearized equation

$$\Delta u_e = W''(u)u_e.$$ 

Lemma

If the Morse index is $N$, the number of connected components of $\{u_e \neq 0\}$ is at most $2N$. 

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**Lemma**

*If the Morse index is $N$, the number of connected components of $\{u_e \neq 0\}$ is at most $2N$.*

- Liouville theorem for the degenerate equation

$$\text{div} \left( \varphi^2 \nabla \frac{u_e}{\varphi} \right) = 0.$$ 

- Similar to Courant’s nodal domain theorem: entire space? $n = 2 \implies$ log cut-off functions, Ambrosio-Cabré ’03...
If each end of \( \{ u = 0 \} \) has an asymptotic direction at infinity, finiteness of nodal domains of \( u_e \) can be transformed into finiteness of ends.
Theorem (W.-Wei '17)

Let $u$ be a finite Morse index solution of the Allen-Cahn equation in $\mathbb{R}^2$. For all $x$ large,

$$|A(x)|^2 := \frac{|\nabla^2 u(x)|^2 - |\nabla|\nabla u(x)||^2}{|\nabla u(x)|^2} \leq \frac{C}{|x|^2}.$$
Curvature decay

**Theorem (W.-Wei ’17)**

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**Theorem (Schoen ’83)**

Given a three dimensional manifold $M$ (with some curvature bounds). Let $\Sigma$ be a stable immersed minimal surface in a ball $B_R(p) \subset M$ with $\partial \Sigma \subset \partial B_R(p)$. Then

$$\sup_{B_{R/2}(p) \cap \Sigma} |A_{\Sigma}|^2 \leq \frac{C}{R^2}.$$
Stability $\Leftrightarrow \int |\nabla \varphi|^2|\nabla u|^2 \geq \int \varphi^2 \left[|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right].$

- $|\nabla u|^2 \, dx$ corresponds to the area measure of minimal surfaces.
Sternberg-Zumbrun inequality

\begin{align*}
\text{Stability} \iff \int |\nabla \varphi|^2 |\nabla u|^2 &\geq \int \varphi^2 \left[ |\nabla^2 u|^2 - |\nabla |\nabla u||^2 \right].
\end{align*}

- $|\nabla u|^2 \, dx$ corresponds to the area measure of minimal surfaces.

- If $|\nabla u| \neq 0$,

\[
\frac{|\nabla^2 u|^2 - |\nabla |\nabla u||^2}{|\nabla u|^2} = |A|^2 + |\nabla T \log |\nabla u||^2,
\]

where $A$ is the second fundamental form of level sets $\{u = \text{const.}\}$ and $\nabla T$ is the tangential derivatives along these level sets.
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where $A$ is the second fundamental form of level sets $\{u = \text{const.}\}$ and $\nabla_T$ is the tangential derivatives along these level sets.

- Simons inequality for this curvature term? Not found yet. Seems to be a common difficulty in semilinear problems.
A blow up proof

- Stability outside $B_R(0) \iff$ Finite Morse index.
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- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$. 

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Finite Morse index solutions

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A blow up proof

- Stability outside $B_R(0) \iff$ Finite Morse index.
- Assume $\exists x_k \in B_R(0)^c$ such that $|A(x_k)||x_k| \geq k$.
- Find $y_k$ satisfying

\[
|A(y_k)| \geq |A(x_k)|, \quad |A(y_k)||y_k| \geq k,
\]

\[
|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_k|A(y_k)|^{-1}(y_k).
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$$|A(x)| \leq 2|A(y_k)|, \quad \forall x \in B_{k|A(y_k)|^{-1}(y_k)}.$$

- Let $\varepsilon_k := |A(y_k)|$ and define $u_k(x) := u(y_k + \varepsilon_k^{-1}x)$. $|y_k| \to +\infty$ and $\varepsilon_k \to 0 \iff$ Locally close to 1D solution, by stable De Giorgi for $n = 2$. 
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- In $B_k(0)$, $u_k$ is a stable solution of (AC) with parameter $\varepsilon_k$. 

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- In $B_k(0)$, $u_k$ is a stable solution of (AC) with parameter $\varepsilon_k$.
- The curvature of $\{u_{\varepsilon_k} = 0\}$ is uniformly bounded, and it equals 1 at the origin.
Second order regularity

**Theorem (W.-Wei ’18)**

Let $u_\varepsilon$ be a sequence of stable solutions to (AC) such that $\{u_\varepsilon = 0\}$ are uniformly $C^{1,\beta}$ for some $\beta \in (0, 1)$. If $n \leq 10$, then $\{u_\varepsilon = 0\}$ are uniformly bounded in $C^{2,\alpha}$ for any $\alpha \in (0, 1)$. Moreover, the mean curvature is of the order $O(\varepsilon^\alpha)$. 
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Chodosh-Mantoulidis ’18 has obtained the same result in dimension 3, which was used in their study of min-max minimal surfaces in three manifolds (Multiplicity one conjecture of Marques-Neves, existence of infinitely many minimal surfaces in generic case).
There could be more and more connected components of \( \{ u_\varepsilon = 0 \} \), which can collapse to the same limit. \( \iff \) clustering interfaces.
Toda system

Curvature bound on $\{u_\varepsilon = 0\} \implies \{u_\varepsilon = 0\}$ locally represented by graphs $\bigcup_k \Gamma_{k,\varepsilon}$, where

$$\Gamma_{k,\varepsilon} = \{x_2 = f_{k,\varepsilon}(x_1)\}, \quad \cdots < f_{k-1,\varepsilon} < f_{k,\varepsilon} < f_{k+1,\varepsilon} < \cdots.$$ 

The cardinality of index set could go to infinity.

Interaction between different interfaces has the form

$$\text{div} \left( \nabla f_k,\varepsilon \sqrt{1 + |\nabla f_k,\varepsilon|^2} \right) = A_\varepsilon \left[ e^{-\sqrt{2} \varepsilon (f_k,\varepsilon - f_{k-1,\varepsilon})} - e^{-\sqrt{2} \varepsilon (f_{k+1,\varepsilon} - f_k,\varepsilon)} \right] + h.$$
Toda system

Curvature bound on \( \{ u_\epsilon = 0 \} \Rightarrow \{ u_\epsilon = 0 \} \) locally represented by graphs \( \bigcup_k \Gamma_{k,\epsilon} \), where

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\[
\text{div} \left( \frac{\nabla f_{k,\epsilon}}{\sqrt{1 + |\nabla f_{k,\epsilon}|^2}} \right) = \frac{A}{\epsilon} \left[ e^{-\frac{\sqrt{2}}{\epsilon} (f_{k,\epsilon} - f_{k-1,\epsilon})} - e^{-\frac{\sqrt{2}}{\epsilon} (f_{k+1,\epsilon} - f_{k,\epsilon})} \right] + h.o.t.
\]

Infinite dimensional Lyapunov-Schmidt reduction of Del Pino, Kowalczyk, Wei and their collaborators.
Interaction between transition layers

Approximate solution: near $\Gamma_k$,

$$g_* := g_k + \sum_{\ell<k} \left[ g_\ell - (-1)^\ell \right] + \sum_{\ell>k} \left[ g_\ell + (-1)^\ell \right].$$
Obstruction to $C^{2,\alpha}$ estimates of $f_{k,\varepsilon}$

\[
\Delta f_{k,\varepsilon} = \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon} (f_{k,\varepsilon} - f_{k-1,\varepsilon})} - \frac{A}{\varepsilon} e^{-\frac{\sqrt{2}}{\varepsilon} (f_{k+1,\varepsilon} - f_{k,\varepsilon})}.
\]

\[
f_{k+1,\varepsilon} - f_{k,\varepsilon} \geq \frac{\sqrt{2} (1 + \alpha)}{2} \varepsilon |\log \varepsilon| - C\varepsilon \implies f_{k,\varepsilon} \in C^{2,\alpha}.
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Obstruction to $C^{2,\alpha}$ estimates of $f_k,\varepsilon$

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\[ f_{k+1,\varepsilon} - f_{k,\varepsilon} \geq \frac{\sqrt{2} (1 + \alpha)}{2} \varepsilon |\log \varepsilon| - C\varepsilon \implies f_{k,\varepsilon} \in C^{2,\alpha}. \]

On the other hand, if

\[ f_{k+1,\varepsilon} - f_{k,\varepsilon} \leq \frac{\sqrt{2}}{2} \varepsilon |\log \varepsilon| + C\varepsilon, \]

define the blow up sequence

\[ \tilde{f}_{k,\varepsilon}(x) := \frac{1}{\varepsilon} f_{k,\varepsilon} \left( \varepsilon^{\frac{1}{2}} x \right) - \frac{\sqrt{2} \alpha}{2} |\log \varepsilon|. \]

They converge to an entire solution of the Toda system

\[ \Delta f_k = e^{-\sqrt{2}(f_k-f_{k-1})} - e^{-\sqrt{2}(f_{k+1}-f_k)}, \quad \text{in } \mathbb{R}^{n-1}. \]
If $u_\varepsilon$ is stable, $(f_k, \varepsilon)$ satisfies a stability condition:

$$\sum_k \int |\nabla \eta_k|^2 \geq \frac{\sqrt{2}A}{\varepsilon^2} \sum_k \int (\eta_k - \eta_{k-1})^2 e^{-\frac{\sqrt{2}}{\varepsilon} (f_k, \varepsilon - f_{k-1}, \varepsilon)} - h.o.t..$$
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$$

Uniform $C^{2,\alpha}$ estimates of clustering interfaces for stable solutions of (AC)
Reduction of the stability condition

- If \( u_\varepsilon \) is stable, \( (f_k, \varepsilon) \) satisfies a stability condition:

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\]

- Uniform \( C^{2,\alpha} \) estimates of clustering interfaces for stable solutions of (AC)

\[\iff\text{Non-existence of entire stable solutions of Toda system.}\]
Liouville theorems for Toda system

\[ \Delta f_k = e^{-\sqrt{2}(f_k-f_{k-1})} - e^{-\sqrt{2}(f_{k+1}-f_k)} \]

Toda system if \( k = 1, \cdots, Q \), Toda lattice if \( k \) runs over \( \mathbb{Z} \).

**Theorem (W. ’19)**

- If \( n < 10 \), there does not exist entire stable solution of Toda lattice in \( \mathbb{R}^n \).
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**Theorem (W. '19)**

- If \( n < 10 \), there does not exist entire stable solution of Toda lattice in \( \mathbb{R}^n \).
- If \( 1 \leq n < 4Q(Q-1)+2 \), there is no entire stable solution of \( Q \)-component Toda system in \( \mathbb{R}^n \).
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- If \( 1 \leq n < 4Q(Q-1)+2 \), there is no entire stable solution of \( Q \)-component Toda system in \( \mathbb{R}^n \).
- If \( 3 \leq n \leq 9 \), there is no solution of Toda lattice in \( \mathbb{R}^n \), which is stable outside a compact set.
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- If \( 1 \leq n < 4Q(Q - 1) + 2 \), there is no entire stable solution of \( Q \)-component Toda system in \( \mathbb{R}^n \).
- If \( 3 \leq n \leq 9 \), there is no solution of Toda lattice in \( \mathbb{R}^n \), which is stable outside a compact set.
- If \( 3 \leq n < N_Q \), there is no finite Morse index solution of \( Q \)-component Toda system in \( \mathbb{R}^n \).
An $\varepsilon$-regularity theorem

**Theorem (W. '19)**

For any $n$, there exists a universal constant $\eta$ such that, if $(f_k)$ is a stable solution to the Toda lattice

$$\Delta f_k = e^{-\sqrt{2}(f_k-f_{k-1})} - e^{-\sqrt{2}(f_{k+1}-f_k)} \text{ in } B_1 \subset \mathbb{R}^n,$$

then

$$\int_{B_1} e^{-\sqrt{2}(f_k-f_{k-1})} \leq \eta(n) \implies \sup_{B_{1/2}} e^{-\sqrt{2}(f_k-f_{k-1})} \leq \frac{1}{2}.$$
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then

$$\int_{B_1} e^{-\sqrt{2}(f_k-f_{k-1})} \leq \eta(n) \quad \implies \quad \sup_{B_{1/2}} e^{-\sqrt{2}(f_k-f_{k-1})} \leq \frac{1}{2}.$$ 

Applying this $\varepsilon$-regularity to suitable rescalings of Toda system constructed from (AC), gives a decay estimate on $e^{-\frac{\sqrt{2}}{\varepsilon}(f_k,\varepsilon-f_{k-1},\varepsilon)}$ in shrinking balls, leading finally to

$$e^{-\frac{\sqrt{2}}{\varepsilon}(f_k,\varepsilon-f_{k-1},\varepsilon)} \lesssim \varepsilon^{1+\alpha}, \quad \text{in the interior.}$$
Prototype results

All results on Toda system are based on corresponding results on the Liouville equation

\[-\Delta u = e^u, \quad \text{in } \mathbb{R}^n.\]

- **Farina '07**: There does not exist stable entire solution if \( n \leq 9 \);
- **Dancer-Farina '09**: There does not exist finite Morse index entire solution if \( 3 \leq n \leq 9 \);
- **W. '12**: \( \varepsilon \)-regularity for stable solutions of Liouville equation.
Denote points in $\mathbb{R}^{n+1}$ by $(x_1, \cdots, x_n, z)$ and let $r := \sqrt{x_1^2 + \cdots + x_n^2}$. A function $u$ is axially symmetric if $u(x_1, \cdots, x_n, z) = u(r, z)$.

**Theorem (Gui-Wei-W. ’19)**

If $3 \leq n \leq 9$, any axially symmetric solution of (AC) in $\mathbb{R}^{n+1}$, which is stable outside a cylinder $C_R$, depends only on $z$, i.e. it is a 1D solution in the $z$-direction.
Theorem (Gui-Wei-W. ’19)

An axially symmetric solution of (AC) in $\mathbb{R}^3$ with finite Morse index has finitely many ends and quadratic energy growth

$$
\int_{B_R(x)} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq CR^2.
$$

(1)
The critical 3D case

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**Theorem (Gui-Wei-W. ’19)**

Suppose $u$ is an axially symmetric solution of (AC) in $\mathbb{R}^3$.

- If it is stable, it depends only on $z \iff$ the solution has one end.
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Theorem (Gui-Wei-W. '19)

Suppose $u$ is an axially symmetric solution of (AC) in $\mathbb{R}^3$.

- If it is stable, it depends only on $z \iff$ the solution has one end.
- If its Morse index equals 1, it has exactly two ends.
The critical 3D case

Theorem (Gui-Wei-W. '19)

An axially symmetric solution of (AC) in $\mathbb{R}^3$ with finite Morse index has finitely many ends and quadratic energy growth

$$\int_{B_R(x)} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \leq CR^2.$$  \hfill (1)

Theorem (Gui-Wei-W. '19)

Suppose $u$ is an axially symmetric solution of (AC) in $\mathbb{R}^3$.

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The structure and moduli space of two-end solutions in $\mathbb{R}^3$ have been studied in detail by Gui-Liu-Wei '17 $\Rightarrow$ Catenoid or Liouville equation.
Thanks for your attention!