Symmetric minimal surfaces in S^3 as global constrained Willmore minimizer in S^n

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Fujian Normal University

几何分析与非线性偏微分方程研讨会,

HIT, May 3, 2019

1. Introduction

2. Minimal surface in S^n and its spectrum properties
   - Minimal surfaces in S^n and first eigenvalue problem
   - Clifford torus
   - Lawson's minimal surfaces $\xi_{m,k}$

3. On Willmore conjecture for higher genus surfaces
   - Symmetric minimal surfaces as constrained Willmore minimizer
   - Li-Yau's conformal area and related results
   - Idea of proof

Willmore functional and Willmore surfaces

- For a closed surface $y : M \to S^n$, the Willmore energy is defined by

$$W(y) := \int_M (|\vec{H}|^2 + 1)\mathrm{d}M.$$

- Willmore conjecture (1965): If $M^2 = T^2$, then $W(y) \geqslant 2\pi^2$, "=" $\Leftrightarrow$ iff f is conformally congruent to the Clifford torus.
- Kusner-Willmore conjecture (1989): If $genus(M^2) = m \geqslant 1$, then $W(y) \geqslant Area(\xi_{m,1})$, with equality iff y is conformally congruent to $\xi_{m,1}$.

  Here $\xi_{m,1}$ is one of simplest Lawson embedded minimal surface with genus m and $Area(\xi_{m,1}) < 8\pi$.
- $\xi_{1,1}$ =Clifford torus.

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Willmore conjecture in S^n

- **Theorem** (Marques & Neves, 2012) If $genus(M^2) \geqslant 1$ and $n = 3$, then $W(y) \geqslant 2\pi^2$, with equality iff y is conformally congruent to the Clifford torus.

Let $T^2(a,b) = \mathbb{R}^2/\Lambda$, with $\Lambda = 2\pi\mathbb{Z} + 2\pi(a+bi)\mathbb{Z}$, $a^2+b^2 \geqslant 1$ and $0 \leqslant a \leqslant 1/2,\ 0 < b$.

- **Theorem** (Li-Yau, 1982) If y is a conformal immersion from $T^2(a,b)$ to S^n with $b \leqslant 1$, then $W(y) \geqslant 2\pi^2$.
- **Theorem** (Montiel-Ros, 1986) If y is a conformal immersion from $T^2(a,b)$ to S^n with $(a-1/2)^2 + (b-1)^2 \leqslant 1/4$, then $W(y) \geqslant 2\pi^2$.

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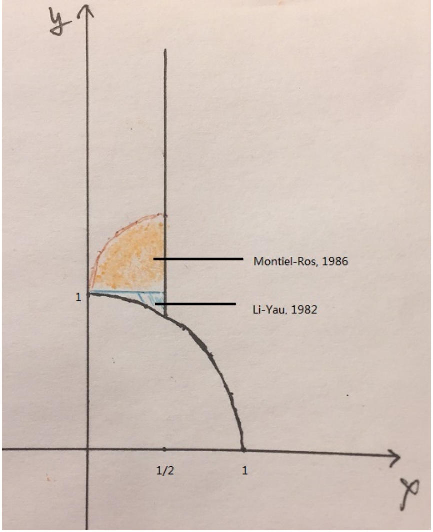
1. Li-Yau and Montiel-Ros's proof on Willmore conjecture for tori in S^n with given conformal structures.



Minimal surfaces in S^n and first eigenvalue of Laplacian

- Let M be a closed Riemann surface with a conformal metric. The eigenvalues of Δ_M are discrete and are tending to $+\infty$:

  $Spec(\Delta_M) = \{0, \lambda_1, \cdots, \}$ and $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$

  λ_1 the first (non-zero) eigenvalue of Δ_M.

- The surface $y: M \to S^n$ is minimal if and only if

$$\Delta_M y = -2y,$$

  i.e., the coordinate functions $y_j, j = 1, \cdots, n+1$, are eigenfunctions of Δ_M with eigenvalue $\lambda = 2$.

- y is called immersed by the first eigenfunctions (of the Laplacian) if $\{y_j\}$ are eigenfunctions of λ_1, i.e., $\lambda_1 = 2$.

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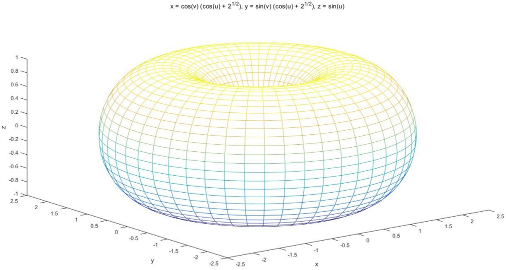
Clifford torus

- $S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$. $T^2 \subset U(2)$ actions on S^3. The orbits of T^2 with maximal area—Clifford torus.
- 

- $Index(T^2) = 5$ in S^3.
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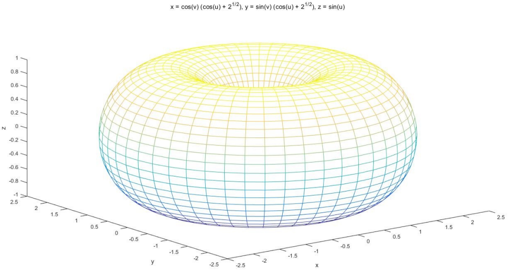
- 

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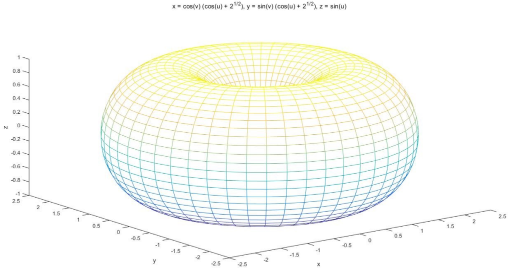
- 

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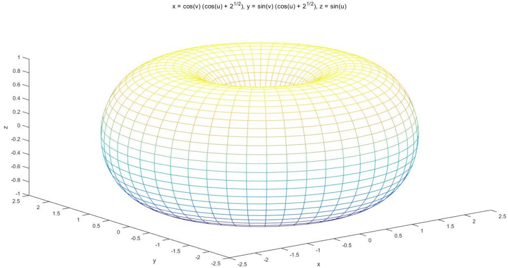
- 

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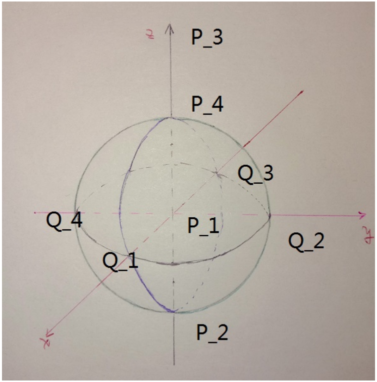
Clifford torus–2

- $y = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v)$,
- $Ay =$
$\left(\cos\frac{u+v}{2}\cos\frac{u-v}{2}, \sin\frac{u+v}{2}\sin\frac{u-v}{2}, \cos\frac{u+v}{2}\sin\frac{u-v}{2}, \sin\frac{u+v}{2}\cos\frac{u-v}{2}\right)$.



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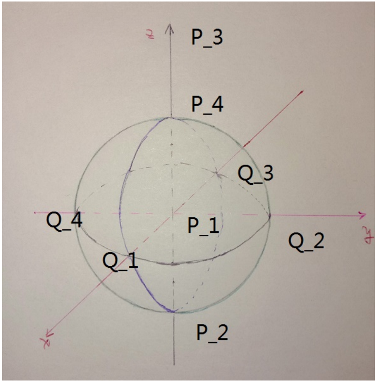
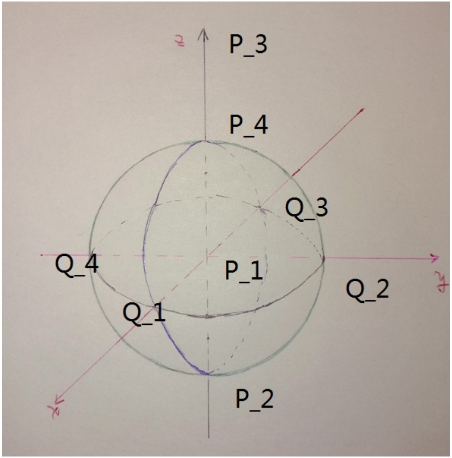
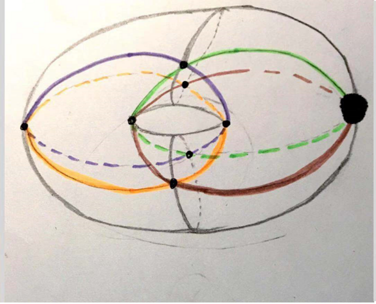
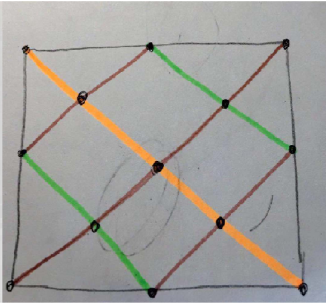


Lawson's minimal surfaces $\xi_{m,k}$

- Lawson's minimal surfaces $\xi_{m,k}$: By reflections w.r.t. geodesics for a solution of Plateau problem.

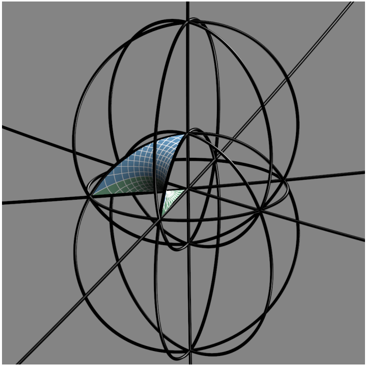


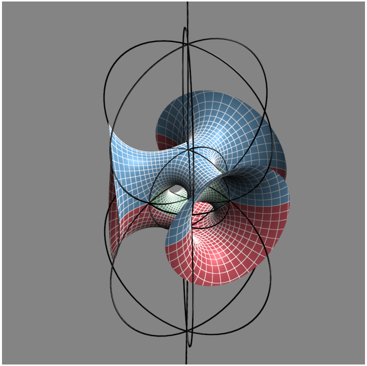




Lawson $\xi_{2,2}$ minimal surfaces (By Nick Schmitt)

https://www.math.uni-tuebingen.de/user/nick/lawson22/





Left: Standard view, cut away by a geodesic 2-sphere.

Right: One of the 9 isometric Plateau solutions which compose the surface. The Plateau solution is the minimal surface bounded by four edges of a geodesic tetrahedron which tiles S^3.

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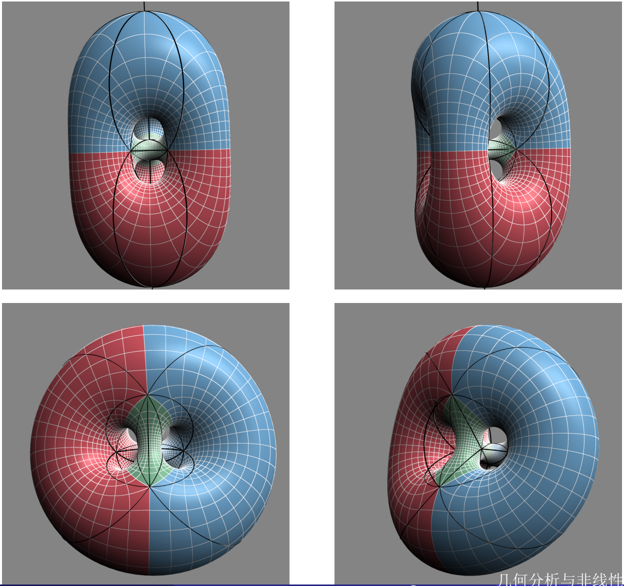

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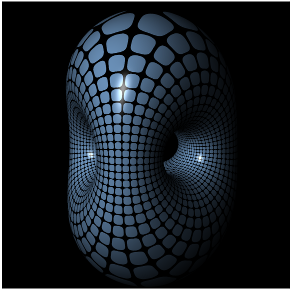
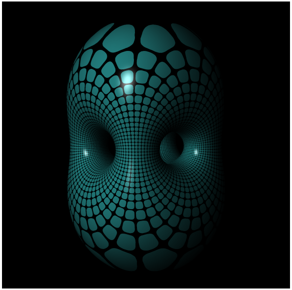
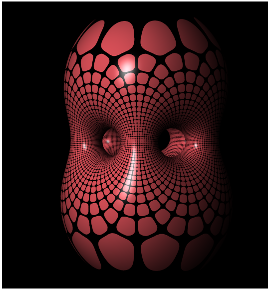
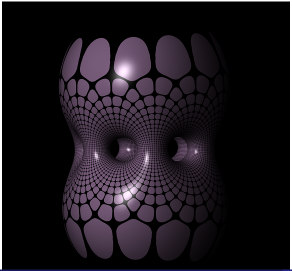
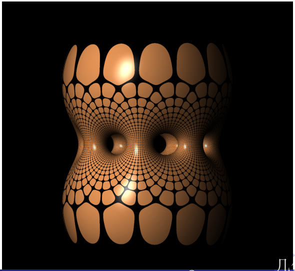
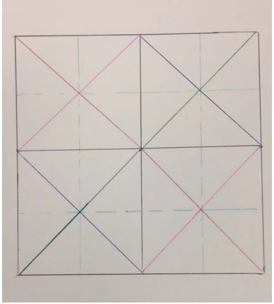






Lawson's minimal surfaces $\xi_{m,k}$

- Karcher-Pinkall-Sterling's examples: By reflections w.r.t. great spheres for a solution of Plateau problem.
- Karcher-Pinkall-Sterling, Choe-Soret: The Lawson minimal surfaces $\xi_{m,k}$ are also symmetric w.r.t. some reflections.



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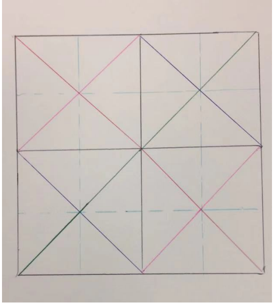

Main result

- **Theorem (Kusner-W, 2018).** Let $\phi : M \to S^3$ be one of the conformal embedded minimal surfaces constructed by Lawson and by Karcher–Pinkall–Sterling. Then for any branched conformal immersion $\tilde{\phi} : M \to S^n$, $n \geqslant 3$,

$$W(\tilde{\phi}) \geqslant W(\phi) = A(\phi).$$

Moreover, " $=$ " $\iff \tilde{\phi}$ is conformally equivalent to ϕ.

Li-Yau's conformal area

Let $\phi : M^2 \to S^n$ be a conformal branched immersion. $Conf(S^n)$ is the conformal group of S^n.

- The *conformal area of* ϕ

$$A_C(n, \phi) := \sup_{\mathbb{T} \in Conf(S^n)} A(\mathbb{T} \circ \phi).$$

  Here $A(\mathbb{T} \circ \phi)$ denotes the area of $\mathbb{T} \circ \phi$.

- The $n-$*conformal area of* M

$$A_C(n, M) := \inf_{\phi} A_C(n, \phi),$$

  where ϕ runs over all conformal branched immersions.

- The *conformal area of* M *is*

  $A_C(M) := \inf_{n \geqslant 2} A_C(n, M) = \lim_{n \to \infty} A_C(n, M).$

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Theorem (Li-Yau,1982) Let $\phi : M \to S^n$ be a branched conformal immersion from a closed Riemann surface. Then

1. The $n-$conformal area satisfies

$$A_C(n, M) \geqslant \frac{1}{2}\lambda_1(M)A(M). \tag{2.1}$$

   Here $A(M)$ is the area of M and $\lambda_1(M)$ is the first (non-zero) eigenvalue of the Laplacian of the metric ds^2.

2. " $=$ " $\Leftrightarrow \exists$ a minimal immersion $\psi : M \to S^{\tilde{n}}$ immersed by the first eigenfunctions, & $A_C(M) = A_C(n, M) = A(\psi)$.

3. The Willmore energy of ϕ

$$W(\phi) = \int_M (H^2 + 1)dM \geqslant A_C(n, M) \geqslant A_C(M). \tag{2.2}$$

   " $=$ " $\Leftrightarrow \phi$ is conformally congruent to a minimal immersion in S^n.

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- **Theorem (Montiel & Ros, 1986; Hirsch & Mäder-Baumdicker, 2017)**: Let $\phi : M \to S^n$ be a minimal surface such that $A_C(n, M) = A(\phi)$. If there exists another conformal minimal immersion $\hat{\phi} : M \to S^{\tilde{n}}$ which is immersed by the first eigenfunctions. Then ϕ is isometric to $\hat{\phi}$. In particular, ϕ is also immersed by the first eigenfunctions.

Idea of proof–1

- **Theorem (Choe & Soret, 2009)**: Let $\phi : M \to S^3$ be one of the embedded minimal surfaces constructed by Lawson and by Karcher–Pinkall–Sterling. Then $\lambda_1(\phi) = 2$.
- **Theorem (Kusner-W, 2018)**: Let $\phi : M \to S^3$ be one of the embedded minimal surfaces constructed by Lawson and by Karcher–Pinkall–Sterling. Then $\dim E_{\lambda_1(\phi)} = 4$.

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Idea of proof–2

- Let G be the finite group generated by reflections of ϕ among the symmetric hyper-spheres γ_j.
- If f is the first eigenfunction of ϕ, then f is $G-$symmetric, i.e.

$$\gamma_j \circ f = f.$$

- If f is $G-$symmetric and orthogonal to the coordinate functions ϕ_j, then $f \equiv 0$.

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The tori case

- **Theorem (Montiel-Ros, 1986).** Let $\phi : T^2(a,b) \to S^n$, $n \geqslant 5$, be a branched conformal immersion with $(a - \frac{1}{2})^2 + (b-1)^2 \leqslant \frac{1}{4}$. Then $W(\phi) \geqslant 2\pi^2$.
- **Theorem (Bryant, 2015).** If $(a - \frac{1}{2})^2 + b^2 \leqslant \frac{9}{4}$, then

$$A_C(T^2(a,b)) = \frac{4\pi^2}{b^2 + a^2 - a + 1}.$$

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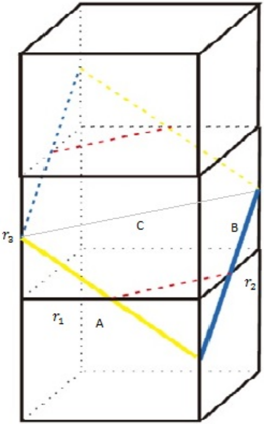
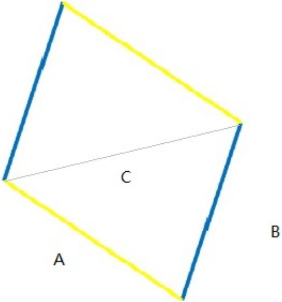


r_3
C
B
r_2
r_1
A



C
B
A

Let $T^2 = \mathbb{C}/\Lambda$ with Λ generated by 1 and $\tau = a + ib$, with $0 \leqslant a \leqslant 1/2$, $b \geqslant \sqrt{1-a^2}$. Then

$$f_\tau(u,v) = \left(\ r_1 e^{i\frac{2\pi v}{b}},\ r_2 e^{i2\pi(u-\frac{va}{b})},\ r_3 e^{i2\pi(u-\frac{v(1-a)}{b})}\ \right). \tag{3.1}$$

with $r_1 = \sqrt{\frac{b^2+a^2-a}{b^2+a^2-a+1}}$, $r_2 = \sqrt{\frac{1-a}{b^2+a^2-a+1}}$, $r_3 = \sqrt{\frac{a}{b^2+a^2-a+1}}$.

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Thank you for your attention!