Symmetric minimal surfaces in $S^3$ as global constrained Willmore minimizer in $S^n$

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1 Introduction

2 Minimal surface in $S^n$ and its spectrum properties
   - Minimal surfaces in $S^n$ and first eigenvalue problem
   - Clifford torus
   - Lawson’s minimal surfaces $\xi_{m,k}$

3 On Willmore conjecture for higher genus surfaces
   - Symmetric minimal surfaces as constrained Willmore minimizer
   - Li-Yau’s conformal area and related results
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Willmore functional and Willmore surfaces

- For a closed surface $y : M \to S^n$, the Willmore energy is defined by

$$W(y) := \int_M (|\vec{H}|^2 + 1)dM.$$ 

- Willmore conjecture (1965): If $M^2 = T^2$, then $W(y) \geq 2\pi^2$, “=” $\iff$ iff $f$ is conformally congruent to the Clifford torus.

- Kusner-Willmore conjecture (1989): If $\text{genus}(M^2) = m \geq 1$, then $W(y) \geq \text{Area}(\xi_{m,1})$, with equality iff $y$ is conformally congruent to $\xi_{m,1}$.

Here $\xi_{m,1}$ is one of simplest Lawson embedded minimal surface with genus $m$ and $\text{Area}(\xi_{m,1}) < 8\pi$.

- $\xi_{1,1} =$ Clifford torus.
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Willmore conjecture in $S^n$

- **Theorem** (Marques & Neves, 2012) If $\text{genus}(M^2) \geq 1$ and $n = 3$, then $W(y) \geq 2\pi^2$, with equality iff $y$ is conformally congruent to the Clifford torus.

Let $T^2(a, b) = \mathbb{R}^2/\Lambda$, with $\Lambda = 2\pi \mathbb{Z} + 2\pi(a + bi)\mathbb{Z}$, $a^2 + b^2 \geq 1$ and $0 \leq a \leq 1/2$, $0 < b$.

- **Theorem** (Li-Yau, 1982) If $y$ is a conformal immersion from $T^2(a, b)$ to $S^n$ with $b \leq 1$, then $W(y) \geq 2\pi^2$.

- **Theorem** (Montiel-Ros, 1986) If $y$ is a conformal immersion from $T^2(a, b)$ to $S^n$ with $(a - 1/2)^2 + (b - 1)^2 \leq 1/4$, then $W(y) \geq 2\pi^2$. 

Peng Wang (Fujian Normal University)  Symmetric minimal surfaces in $S^3$ as global constrained Willmore minimizer in $S^n$
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Li-Yau and Montiel-Ros’s proof on Willmore conjecture for tori in $S^n$ with given conformal structures.

![Diagram of symmetric minimal surfaces in $S^3$ as global constrained Willmore minimizer in $S^n$](image)
Let $M$ be a closed Riemann surface with a conformal metric. The eigenvalues of $\Delta_M$ are discrete and are tending to $+\infty$:

$$\text{Spec}(\Delta_M) = \{0, \lambda_1, \cdots , \}$$

and $0 < \lambda_1 \leq \lambda_2 \leq \cdots$

$\lambda_1$ the first (non-zero) eigenvalue of $\Delta_M$.

The surface $y : M \to S^n$ is minimal if and only if

$$\Delta_M y = -2y,$$

i.e., the coordinate functions $y_j, j = 1, \cdots , n + 1$, are eigenfunctions of $\Delta_M$ with eigenvalue $\lambda = 2$.

$y$ is called immersed by the first eigenfunctions (of the Laplacian) if \{y_j\} are eigenfunctions of $\lambda_1$, i.e., $\lambda_1 = 2$. 

Minimal surfaces in $S^n$ and first eigenvalue of Laplacian

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Clifford torus

- $S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$. $T^2 \subset U(2)$ actions on $S^3$. The orbits of $T^2$ with maximal area—Clifford torus.

- $\text{Index}(T^2) = 5$ in $S^3$.
- $\text{Index}(T^2) = 1 + (n + 1) = n + 2$ in $S^n$. 
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x = \cos(v)(\cos(u) + z^2), \quad y = \sin(v)(\cos(u) + z^2), \quad z = \sin(u)
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Clifford torus–2

- \( y = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v), \)
- \( Ay = \)
  \[
  \left( \cos \frac{u+v}{2} \cos \frac{u-v}{2}, \sin \frac{u+v}{2} \sin \frac{u-v}{2}, \cos \frac{u+v}{2} \sin \frac{u-v}{2}, \sin \frac{u+v}{2} \cos \frac{u-v}{2} \right).
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Lawson $\xi_{2,2}$ minimal surfaces (By Nick Schmitt)

https://www.math.uni-tuebingen.de/user/nick/lawson22/

Left: Standard view, cut away by a geodesic 2-sphere.
Right: One of the 9 isometric Plateau solutions which compose the surface. The Plateau solution is the minimal surface bounded by four edges of a geodesic tetrahedron which tiles $S^3$. 
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- Karcher-Pinkall-Sterling’s examples: By reflections w.r.t. great spheres for a solution of Plateau problem.

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Theorem (Kusner-W, 2018). Let \( \phi : M \to S^3 \) be one of the conformal embedded minimal surfaces constructed by Lawson and by Karcher–Pinkall–Sterling. Then for any branched conformal immersion \( \tilde{\phi} : M \to S^n, \ n \geq 3, \)

\[
W(\tilde{\phi}) \geq W(\phi) = A(\phi).
\]

Moreover, “=” \( \iff \tilde{\phi} \) is conformally equivalent to \( \phi \).
Li-Yau’s conformal area

Let $\phi : M^2 \rightarrow S^n$ be a conformal branched immersion. $Conf(S^n)$ is the conformal group of $S^n$.

- The conformal area of $\phi$

$$A_C(n, \phi) := \sup_{T \in Conf(S^n)} A(T \circ \phi).$$

Here $A(T \circ \phi)$ denotes the area of $T \circ \phi$.

- The $n$–conformal area of $M$

$$A_C(n, M) := \inf_{\phi} A_C(n, \phi),$$

where $\phi$ runs over all conformal branched immersions.

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Theorem (Li-Yau, 1982) Let $\phi : M \to S^n$ be a branched conformal immersion from a closed Riemann surface. Then

1. The $n-$conformal area satisfies

$$A_C(n, M) \geq \frac{1}{2} \lambda_1(M) A(M). \quad (2.1)$$

Here $A(M)$ is the area of $M$ and $\lambda_1(M)$ is the first (non-zero) eigenvalue of the Laplacian of the metric $ds^2$.

2. "\(=\)" $\iff \exists$ a minimal immersion $\psi : M \to S^{\tilde{n}}$ immersed by the first eigenfunctions, & $A_C(M) = A_C(n, M) = A(\psi)$.

3. The Willmore energy of $\phi$

$$W(\phi) = \int_M (H^2 + 1) dM \geq A_C(n, M) \geq A_C(M). \quad (2.2)$$

"\(=\)" $\iff \phi$ is conformally congruent to a minimal immersion in $S^n$. 
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Theorem (Montiel & Ros, 1986; Hirsch & Mäder-Baumdicker, 2017): Let \( \phi : M \to S^n \) be a minimal surface such that \( A_C(n, M) = A(\phi) \). If there exists another conformal minimal immersion \( \hat{\phi} : M \to S^{\tilde{n}} \) which is immersed by the first eigenfunctions. Then \( \phi \) is isometric to \( \hat{\phi} \). In particular, \( \phi \) is also immersed by the first eigenfunctions.
Theorem (Choe & Soret, 2009): Let \( \phi : M \to S^3 \) be one of the embedded minimal surfaces constructed by Lawson and by Karcher–Pinkall–Sterling. Then \( \lambda_1(\phi) = 2 \).

Theorem (Kusner-W, 2018): Let \( \phi : M \to S^3 \) be one of the embedded minimal surfaces constructed by Lawson and by Karcher–Pinkall–Sterling. Then \( \dim E_{\lambda_1}(\phi) = 4 \).
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Idea of proof–2

- Let $G$ be the finite group generated by reflections of $\phi$ among the symmetric hyper-spheres $\gamma_j$.

- If $f$ is the first eigenfunction of $\phi$, then $f$ is $G$–symmetric, i.e.

  $$\gamma_j \circ f = f.$$ 

- If $f$ is $G$–symmetric and orthogonal to the coordinate functions $\phi_j$, then $f \equiv 0$. 

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The tori case

- **Theorem (Montiel-Ros, 1986).** Let $\phi : T^2(a, b) \rightarrow S^n, n \geq 5$, be a branched conformal immersion with $(a - \frac{1}{2})^2 + (b - 1)^2 \leq \frac{1}{4}$. Then $W(\phi) \geq 2\pi^2$.

- **Theorem (Bryant, 2015).** If $(a - \frac{1}{2})^2 + b^2 \leq \frac{9}{4}$, then

$$AC(T^2(a, b)) = \frac{4\pi^2}{b^2 + a^2 - a + 1}.$$

- **Theorem (Kusner-W, 2018).** In the above case, $W(\phi) = 2\pi^2$ if and only if $\phi$ is conformally equivalent to the Clifford torus in $S^3$. 
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- **Theorem (Bryant, 2015).** If $(a - \frac{1}{2})^2 + b^2 \leq \frac{9}{4}$, then

  $$A_C(T^2(a, b)) = \frac{4\pi^2}{b^2 + a^2 - a + 1}.$$

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Symmetric minimal surfaces in $S^3$ as global constrained Willmore minimizers.
Let $T^2 = \mathbb{C}/\Lambda$ with $\Lambda$ generated by 1 and $\tau = a + ib$, with $0 \leq a \leq 1/2$, $b \geq \sqrt{1-a^2}$. Then

$$f_\tau(u, v) = \left( r_1 e^{i \frac{2\pi v}{b}}, r_2 e^{i 2\pi (u - \frac{va}{b})}, r_3 e^{i 2\pi (u - \frac{v(1-a)}{b})} \right).$$  \hspace{1cm} (3.1)$$

with $r_1 = \sqrt{\frac{b^2 + a^2 - a}{b^2 + a^2 - a + 1}}$, $r_2 = \sqrt{\frac{1-a}{b^2 + a^2 - a + 1}}$, $r_3 = \sqrt{\frac{a}{b^2 + a^2 - a + 1}}$. 


Thank you for your attention!