Existence and Boundary Asymptotic Behavior of Hessian Equations

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Outline

Introduction

We investigate the following k-Hessian equation $(1 \leq k \leq n)$:

$$\begin{cases} S_k(D^2u) = \sigma_k(\lambda) = b(x)f(-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} S_k(D^2u) = \sigma_k(\lambda) = b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix D^2u and

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k^{th} elementary symmetric function of λ. For completeness, we also set $\sigma_0(\lambda) = 1$ and $\sigma_k(\lambda) = 0$ for $k > n$.

Introduction

To work in the realm of elliptic operators, we have to restrict the class of functions and domains.

$u \in C^2(\Omega)$ is called a k-admissible function if for any $x \in \Omega$, $\lambda(D^2u(x))$ belongs to the cone given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \cdots, k\}.$$

Γ_k is an open, convex, symmetric cone with vertex at the origin, and

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_1, \cdots, \lambda_n > 0\}.$$

We have

$$\frac{\partial \sigma_k(\lambda)}{\partial \lambda_i} > 0 \ \text{ in } \ \Gamma_k \ \ \forall \ i$$

and

$$\sigma_k^{1/k}(\lambda) \text{ is a concave function in } \Gamma_k.$$

Introduction

Let

$$S(\Gamma_k) = \{A : A \in \mathbb{S}^{n\times n}, \lambda(A) \in \Gamma_k\},$$

where $\mathbb{S}^{n\times n}$ denotes the set of $n \times n$ real symmetric matrices. $S(\Gamma_k)$ is an open convex cone with vertex at the origin in matrix spaces. The properties of σ_k described above guarantee that

$$\left(\frac{\partial S_k}{\partial a_{ij}}\right)_{n\times n} > 0 \quad \forall\ A \in S(\Gamma_k)$$

and

$$S_k^{1/k} \text{ is concave in } S(\Gamma_k).$$

Moreover, for any $1 \leq l \leq k$,

$$\left(\frac{\partial S_l}{\partial a_{ij}}\right)_{n\times n} > 0 \quad \forall\ A \in S(\Gamma_k).$$

For convenience, we will denote $S_l^{ij}(A) = \frac{\partial S_l(A)}{\partial a_{ij}}$.

Introduction

For an open bounded subset $\Omega \subset \mathbb{R}^n$ with boundary of class C^2 and for every $x \in \partial\Omega$, we denote by

$$\rho(x) = (\rho_1(x), \cdots, \rho_{n-1}(x))$$

the principal curvatures of $\partial\Omega$ (relative to the interior normal).

Ω is said to be l-convex $(1 \leq l \leq n-1)$ if $\partial\Omega$, regarded as a hypersurface in $\mathbb{R}^n$, is l-convex, that is, for every $x \in \partial\Omega$,

$$\sigma_j(\rho(x)) \geq 0 \quad \text{with} \quad j = 1, 2, \cdots, l.$$

Respectively, Ω is called strictly l-convex if

$$\sigma_j(\rho(x)) > 0 \quad \text{with} \quad j = 1, 2, \cdots, l.$$

Note that $\sigma_0(\rho(x)) = 1 > 0$ for any $x \in \partial\Omega$. We say that every bounded domain Ω with boundary of class C^2 is 0-convex.

Introduction

Definition 1. A function $u \in C(\Omega)$ is said to be a viscosity subsolution (supersolution) of (0.1) if whenever $x_0 \in \Omega$, A is an open neighborhood of x_0, $\psi \in C^2(A)$ is k-admissible and $u - \psi$ has a local maximum (minimum) at x_0, then

$$S_k(D^2\psi(x_0)) \geq b(x_0)f(-\psi(x_0)) \ (\leq b(x_0)f(-\psi(x_0))).$$

A function $u \in C(\Omega)$ is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Definition 1′. A function $u \in C(\Omega)$ is said to be a viscosity subsolution (supersolution) of (0.2) if whenever $x_0 \in \Omega$, A is an open neighborhood of x_0, $\psi \in C^2(A)$ is k-admissible and $u - \psi$ has a local maximum (minimum) at x_0, then

$$S_k(D^2\psi(x_0)) \geq b(x_0)f(\psi(x_0)) \ (\leq b(x_0)f(\psi(x_0))).$$

A function $u \in C(\Omega)$ is said to be a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Existence of solutions

Theorem 1: Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial\Omega \in C^{3,1}$. Suppose that $f \in C^1(0,\infty)$ is positive and nonincreasing, and that $b \in C^{1,1}(\overline{\Omega})$ is positive in Ω. Then (0.1) admits a unique viscosity solution $u \in C(\overline{\Omega})$.

Remark. The main interest here is that of Hessian equations with singular right-hand sides. Note that it may happen that

$$f(s) \to \infty \quad \text{as} \quad s \to 0.$$

The existence of solutions of Hessian equations with regular right-hand sides has been considered in many papers, c.f.:

[1] K.S. Chou, X.J. Wang, A variational theory of the Hessian equation, Comm. Pure Appl. Math. 54 (2001) 1029-1064.

Related results

[2] N.S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995) 151-164.

Lemma 1. Assume that Ω is bounded, strictly (k-1)-convex and $\partial\Omega \in C^{3,1}$. Let $b(x) \in C^{1,1}(\overline{\Omega})$ be positive in Ω. Then the following equation

$$\begin{cases} S_k(D^2u) = b(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a k-admissible solution $u \in C^{3,\beta}_{loc}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \beta < 1$.

Related results

[3] B. Guan, The Dirichlet problem for a class of fully nonlinear elliptic equations, Comm. Partial Differential Equations 19 (1994) 399-416.

[4] Y.Y. Li, Some existence results for fully nonlinear elliptic equations of Monge-Ampère type, Comm. Pure Appl. Math. 43 (1990) 233-271.

Lemma 2. Assume that Ω is bounded, strictly (k-1)-convex and $\partial\Omega \in C^{3,1}$. Let $f \in C^1(0,\infty)$ be positive and nonincreasing, and $b(x) \in C^{1,1}(\overline{\Omega})$ be positive in Ω. For any constant $c < 0$, if

$$\begin{cases} S_k(D^2u) = b(x)f(-u) & \text{in } \Omega, \\ u = c & \text{on } \partial\Omega, \end{cases}$$

has a subsolution, then it admits a solution $u \in C^{3,\beta}_{loc}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \beta < 1$.

The comparison principle

Lemma 3. (The comparison principle.) Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $g(x,\eta)$ is positive and continuously differentiable, and is nondecreasing only with respect to η. If $u, v \in C(\overline{\Omega})$ are respectively viscosity subsolution and supersolution of

$$S_k(D^2u) = g(x,u)$$

and $u \leq v$ on $\partial\Omega$, then we have

$$u \leq v \quad \text{in } \Omega.$$

Moreover, the conclusion is still true if $v \in C(\Omega)$ and $v(x) \to \infty$ as $x \to \partial\Omega$.

J.I.E. Urbas, On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations, Indiana Univ. Math. J. 39 (1990) 355-382.

Proof of Theorem 1.

By Lemma 1, let $w \in C^{3,\beta}_{loc}(\Omega) \cap C(\overline{\Omega})$ $(0 < \beta < 1)$ be the solution of

$$\begin{cases} S_k(D^2 w) = b(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Define

$$\underline{v}(x) = -\eta(-w(x)) \quad \text{in } \Omega,$$

where η is given by

$$t = \int_0^{\eta(t)} \frac{1}{f(\tau)^{1/k}} d\tau.$$

Proof of Theorem 1.

Then we see that

$$\eta(0) = 0, \ \eta'(t) = f(\eta(t))^{1/k} \text{ and } \eta''(t) = \frac{1}{k} f(\eta(t))^{(2-k)/k} f'(\eta(t))$$

and

$$\underline{v}_{ij} = \eta'(-w) w_{ij} - \eta''(-w) w_i w_j.$$

It follows that

$$D^2 \underline{v} \geq \eta'(-w) D^2 w.$$

Therefore $\underline{v}$ is k-admissible and

$$S_k(D^2 \underline{v}) \geq (\eta'(-w))^k S_k(D^2 w) = (\eta'(-w))^k b(x) = b(x) f(-\underline{v}) \text{ in } \Omega.$$

Proof of Theorem 1.

Let

$$\Omega_j = \{x \in \Omega : \underline{v}(x) < -\frac{1}{j}\}$$

for $j = 1, 2, \cdots$. Then Ω_j is strictly $(k-1)$-convex since w is k-admissible. *c.f. N.S. Trudinger, On new isoperimetric inequalities and symmetrization, J. Reine Angew. Math. 488 (1997) 203-220.*

Consider

$$\begin{cases} S_k(D^2 u) = b(x) f(-u) & \text{in } \Omega_j, \\ u = -\frac{1}{j} & \text{on } \partial\Omega_j. \end{cases} \quad (2.1)$$

We have shown $\underline{v}$ is its k-admissible subsolution. Then by Lemma 2, it has a k-admissible solution u_j.

Proof of Theorem 1.

By Lemma 3,

$$\underline{v} \leq u_j \quad \text{in } \Omega_j.$$

Since

$$u_j = \underline{v} \leq u_{j+1} \quad \text{on } \partial\Omega_j,$$

by lemma 3 again,

$$u_j \leq u_{j+1} \quad \text{in } \Omega_j.$$

Proof of Theorem 1.

Let

$$\overline{v} = \alpha w \quad \text{in } \Omega.$$

Then $\overline{v}$ is a supersolution of (2.1) if $\alpha > 0$ is sufficiently small.

Actually, let $L_w = \max\{-w(x) : x \in \overline{\Omega}\}$ and α small enough such that

$$\alpha^k \leq f(\alpha L_w) \quad \text{and} \quad \alpha^k \leq f(1).$$

Therefore $\overline{v}$ satisfies

$$\sigma_k(\lambda(D^2(\overline{v}))) = \alpha^k b(x) \leq b(x) f(\alpha L_w) \leq b(x) f(-\overline{v}) \quad \text{in } \Omega.$$

Proof of Theorem 1.

Note that

$$t \le \frac{\eta(t)}{f(\eta(t))^{1/k}} \quad \text{for} \quad 0 < \eta(t) < 1.$$

Thus

$$\eta^{-1}(s) \le \frac{s}{f(s)^{1/k}} \le \frac{s}{f(1)^{1/k}} \quad \text{for} \quad 0 < s < 1.$$

We see that

$$\overline{v} = \alpha w = -\alpha\eta^{-1}(\frac{1}{j}) \ge -\frac{1}{j}\frac{\alpha}{f(1)^{1/k}} \ge -\frac{1}{j} \quad \text{on } \partial\Omega_j.$$

By Lemma 3,

$$u_j \le \overline{v} \quad \text{in } \Omega_j.$$

Proof of Theorem 1.

For each $x \in \Omega$, choose j_0 so that $x \in \Omega_{j_0}$. For any $j \geq j_0$,

$$\underline{v} \leq u_j \leq u_{j+1} \leq \overline{v} \quad \text{in } \Omega_{j_0}.$$

Let

$$u(x) = \lim_{j\to\infty} u_j(x).$$

Moreover, $b(x)f(-u_j(x)) \in L^\infty(\Omega_j)$. For any $j \geq j_0 + 1$, by

N.S. Trudinger, Weak solutions of Hessian equations, CPDE 22 (1997) 1251-1261

$u_j \in C^\beta(\Omega_{j_0+1})$ with $0 < \beta < 1$ and

$$\|u_j\|_{C^\beta(\overline{\Omega}_{j_0})} \leq C(n, k, \min_{\overline{\Omega}_{j_0}} \underline{v}, \max_{\overline{\Omega}_{j_0}} \overline{v}, \Omega_{j_0}, b, f).$$

Proof of Theorem 1.

Hence $u \in C(\Omega)$. Note that

$$\underline{v} \leq u \leq \overline{v} \quad \text{in } \Omega.$$

This implies that $u \in C(\overline{\Omega})$.

By the uniform convergence of $\{u_j\}$ on compact subset of Ω, we have u is a viscosity solution of (1.1).

By Lemma 3, the solution is unique.

Boundary asymptotic behavior

To investigate the boundary behavior of solutions of (1.1), we need more assumptions on

f,

b and

the curvature of the boundary.

Assumptions on f

($\mathbf{f_1}$) $f \in C^1(0,\infty)$, $f(s) > 0$, $f(s) \to \infty$ as $s \to 0$, and is nonincreasing on $(0,\infty)$;

($\mathbf{f_2}$) There exists $C_f > 0$ such that

$$\lim_{s\to 0^+} H'(s) \int_0^s \frac{d\tau}{H(\tau)} = -C_f,$$

where $H(\tau) = ((k+1)F(\tau))^{1/(k+1)}$ and $F(\tau) = \int_\tau^a f(s)ds \;\; \forall\, 0 < \tau < a$. For convenience, we define φ by

$$\int_0^{\varphi(t)} \frac{d\tau}{H(\tau)} = t \;\; \forall\, 0 < t < \alpha,$$

where $\varphi(\alpha) = a$. Actually, the existence of φ is obvious since $\frac{1}{H}$ is nondecreasing and integrable on $[0, a]$.

Assumptions on b

($\mathbf{b_1}$) $b \in C^{1,1}(\overline{\Omega})$ is positive in Ω;

($\mathbf{b_2}$) There exist a positive and nondecreasing function $m(t) \in C^1(0, \delta_0)$ (for some $\delta_0 > 0$), and two positive constants $\overline{b}$ and $\underline{b}$ such that

$$\underline{b} = \liminf_{\substack{x \in \Omega \\ d(x) \to 0}} \frac{b(x)}{m^{k+1}(d(x))} \leq \limsup_{\substack{x \in \Omega \\ d(x) \to 0}} \frac{b(x)}{m^{k+1}(d(x))} = \overline{b},$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$, and there exists $C_m \in [0, \infty)$ such that

$$\lim_{t \to 0^+} \left(\frac{M(t)}{m(t)} \right)' = C_m,$$

where $M(t) = \int_0^t m(s)ds < \infty$ for any $0 < t < \delta_0$.

Curvatures of the boundary

Set

$$L_0 = \max_{\overline{x}\in\partial\Omega} \sigma_{k-1}(\rho(\overline{x})) \ \text{ and } \ l_0 = \min_{\overline{x}\in\partial\Omega} \sigma_{k-1}(\rho(\overline{x})),$$

where

$$\rho(\overline{x}) = (\rho_1(\overline{x}), \rho_2(\overline{x}), \cdots, \rho_{n-1}(\overline{x}))$$

are the principal curvatures of $\partial\Omega$ at $\overline{x}$. Observe that $0 < l_0 \leq L_0 < +\infty$ since Ω is bounded and strictly $(k-1)$-convex.

The boundary estimates of the solution of (1.1) are related to L_0 and l_0.

Theorem 2.

Theorem 2: Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial\Omega \in C^{3,1}$. Suppose that f satisfies $(\mathbf{f_1})$ and $(\mathbf{f_2})$, and b satisfies $(\mathbf{b_1})$ and $(\mathbf{b_2})$. If

$$C_f > 1 - C_m, \qquad (2.2)$$

then the viscosity solution u of (1.1) satisfies

$$1 \leq \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{-u(x)}{\varphi(\underline{\xi} M(d(x)))} \quad \text{and} \quad \limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{-u(x)}{\varphi(\overline{\xi} M(d(x)))} \leq 1, \qquad (2.3)$$

where φ is defined by (1.3),

$$\underline{\xi} = \left(\frac{\underline{b}}{L_0(1 - C_f^{-1}(1 - C_m))}\right)^{\frac{1}{k+1}}, \quad \overline{\xi} = \left(\frac{\overline{b}}{l_0(1 - C_f^{-1}(1 - C_m))}\right)^{\frac{1}{k+1}}.$$

Example 1

(i) $b \equiv 1$ and $f(s) = s^{-\gamma}$, $\gamma > 1$. Choose $m(t) = 1$ and then

$$M(t) = t \quad \text{and} \quad C_m = 1.$$

We obtain $C_f = \frac{\gamma-1}{k+\gamma} > 1 - C_m$, ((2.2)holds)

$\varphi(t) = \left(\frac{(k+\gamma)^{k+1}}{(\gamma-1)(k+1)^k}\right)^{1/(k+\gamma)} t^{(k+1)/(k+\gamma)}$,

$\underline{\xi} = \left(\frac{1}{L_0}\right)^{1/(k+1)}$ and $\overline{\xi} = \left(\frac{1}{l_0}\right)^{1/(k+1)}$. Hence

$$1 \leq \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{-u(x)}{\left(\frac{(k+\gamma)^{k+1}}{L_0(\gamma-1)(k+1)^k}\right)^{1/(k+\gamma)} d(x)^{(k+1)/(k+\gamma)}}$$

and

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{-u(x)}{\left(\frac{(k+\gamma)^{k+1}}{l_0(\gamma-1)(k+1)^k}\right)^{1/(k+\gamma)} d(x)^{(k+1)/(k+\gamma)}} \leq 1.$$

Example 2

(ii) $b = d(x)^{\alpha(k+1)}$, $\alpha > 0$, near $\partial\Omega$ and $f(s) = s^{-\gamma}$, $\gamma > 1$.
In this case, choose $m(t) = t^{\alpha}$, and we obtain

$$M(t) = \frac{t^{\alpha+1}}{\alpha+1} \quad \text{and} \quad C_m = \frac{1}{\alpha+1}.$$

We still have $C_f = \frac{\gamma-1}{k+\gamma}$,

$$\varphi(t) = \left(\frac{(k+\gamma)^{k+1}}{(\gamma-1)(k+1)^k}\right)^{1/(k+\gamma)} t^{(k+1)/(k+\gamma)},$$

$$\underline{\xi} = \left(\frac{(\alpha+1)(\gamma-1)}{L_0(\gamma-1-\alpha k-\alpha)}\right)^{\frac{1}{k+1}} \text{ and } \overline{\xi} = \left(\frac{(\alpha+1)(\gamma-1)}{l_0(\gamma-1-\alpha k-\alpha)}\right)^{\frac{1}{k+1}}.$$

Example 2

If $\gamma > \alpha(k+1)+1$, then $C_f > 1 - C_m$. ((2.2) holds)

Therefore, by (1.6), the solution of (1.1) satisfies

$$1 \le \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{-u(x)}{\left(\frac{(k+\gamma)^{k+1}}{L_0(\gamma-\alpha k-\alpha-1)(k+1)^k(\alpha+1)^k}\right)^{1/(k+\gamma)} d(x)^{(k+1)(\alpha+1)/(k+\gamma)}}$$

and

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{-u(x)}{\left(\frac{(k+\gamma)^{k+1}}{l_0(\gamma-\alpha k-\alpha-1)(k+1)^k(\alpha+1)^k}\right)^{1/(k+\gamma)} d(x)^{(k+1)(\alpha+1)/(k+\gamma)}} \le 1.$$

Some related results

M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (1977) 193-222.

The existence of classical solution of Poisson equations and the boundary behavior of the solution with f only satisfying $(\mathbf{f_1})$ and $b = 1$. They showed that the unique solution u satisfies

$$c_1 p(d(x)) \leq -u(x) \leq c_2 p(d(x)) \text{ in } \Omega_\alpha,$$

where c_1 and c_2 are two positive constants, $\Omega_\alpha = \{x \in \Omega : d(x) \leq \alpha\}$ for some $\alpha > 0$, and p satisfies

$$\begin{cases} -p''(s) = f(p(s)) & \text{for } 0 < s \leq \alpha, \\ p(0) = 0, & \\ p(s) > 0 & \text{for } 0 < s \leq \alpha. \end{cases}$$

This is actually the generalization of Hopf lemma for singular elliptic equations.

Some related results

A.C. Lazer, P.J. McKenna, On singular boundary value problems for the Monge-Ampère operator, J. Math. Anal. Appl. 197 (1996) 341-362.

Generalize the results to Monge-Ampère equations with $f(s) = s^{-\gamma}, \gamma > 1$ and a positive $b \in C^{\infty}(\overline{\Omega})$. They showed that there exist two positive constants k_1 and k_2 such that

$$k_1 d(x)^{\frac{n+1}{n+\gamma}} \leq -u(x) \leq k_2 d(x)^{\frac{n+1}{n+\gamma}} \text{ in } \Omega.$$

A. Mohammed, Existence and estimates of solutions to a singular Dirichlet problem for the Monge-Ampère equation, J. Math. Anal. Appl. 340 (2008) 1226-1234.

Generalize to f only satisfying $(\mathbf{f_1})$ and the result: there are two positive constants C_1 and C_2 such that

$$C_1 \varphi(d(x)) \leq -u(x) \leq C_2 \varphi(d(x)) \text{ in } \Omega_{\alpha},$$

where φ is defined by (1.3) with k being replaced by n.

Some related results

L and S.S. Ma, Boundary behavior of solutions of Monge-Ampère equations with singular righthand sides, J. Math. Anal. Appl. 454 (2017) 79-93.

H.Y.Sun and M.Q.Feng, Boundary behavior of k-convex solutions for singular k-Hessian equations, Nonlinear Anal. 176 (2018), 141-156.

C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in: Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers), Academic Press, New York, 1974, pp. 245-274.

S.Y. Cheng, S.T. Yau, On the regularity of the Monge-Amp'ere equation $det(\partial^2 u/\partial xi \partial xj) = F(x, u)$, Comm. Pure Appl. Math. 30 (1977) 41.68.

M. Ghergu, V. Radulescu, Singular Elliptic Problems: Bifurcation and Asymptotic Analysis, in: Oxford Lecture Series in Mathematics and its Applications, vol. 37, The Clarendon Press

Proof of Theorem 2.

We will prove Theorem 2 by the comparison principle. The key is to construct supersolution and subsolution.

The functions $-\varphi(\underline{\xi}M(d(x)))$ and $-\varphi(\overline{\xi}M(d(x)))$ in (1.6), are "quasi-supersolution" and "quasi-subsolution" respectively.

That is, after perturbing $\underline{\xi}$ and $\overline{\xi}$ to $\underline{\xi}_\varepsilon$ and $\overline{\xi}_\varepsilon$, $-\varphi(\underline{\xi}_\varepsilon M(d(x)))$ and $-\varphi(\overline{\xi}_\varepsilon M(d(x)))$ are supersolution and subsolution near the boundary respectively.

Two Lemmas

We need the asymptotic estimate of functions in $(\mathbf{f_2})$ and $(\mathbf{b_2})$ as $t \to 0$. The following two lemmas describe those asymptotic behaviors. As for their proofs, Karamata regular variation theory was used. Karamata regular variation theory is a tool to describe the precise rate of functions tending to zero or infinity.

J. Karamata, Sur un mode de croissance régulière. Théorèmes fondamentaux, Bull. Soc. Math. France. 61(1933)55-62.

Lemma 4. Let m and M be the functions given by $(\mathbf{b_2})$. Then

$$\lim_{t\to 0^+} \frac{M(t)}{m(t)} = 0$$

and

$$\lim_{t\to 0^+} \frac{M(t)m'(t)}{m^2(t)} = 1 - C_m.$$

Two Lemmas

Lemma 5. Assume that f satisfies $(\mathbf{f_1})$ and $(\mathbf{f_2})$, and φ satisfies (1.3). Then we have

(i_1) $\varphi(0) = 0, \ \varphi(t) > 0, \ \varphi'(t) = ((k+1)F(\varphi(t)))^{\frac{1}{k+1}}$,

$$\varphi''(t) = -((k+1)F(\varphi(t)))^{(1-k)/(k+1)} f(\varphi(t));$$

(i_2) $\lim_{t\to 0^+} \frac{\varphi'(t)}{t\varphi''(t)} = -\frac{1}{C_f}$;

(i_3) If (1.5) holds,

$$\lim_{t\to 0^+} \frac{t}{\varphi(\xi M(t))} = 0$$

for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.

Properties of distance function

Let $d(x) = \text{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$. For any $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

If $\partial\Omega \in C^2$, there exists $\delta_1 > 0$ such that

$$d \in C^2(\Omega_{\delta_1}).$$

Let $\overline{x} \in \partial\Omega$ be such that $\text{dist}(x, \partial\Omega) = |x - \overline{x}|$ and $\rho_i(\overline{x})(i = 1, \cdots, n-1)$ be the principal curvatures of $\partial\Omega$ at $\overline{x}$. Then, in terms of a principal coordinate system at $\overline{x}$, we have

$$\begin{cases} Dd(x) = (0, 0, \cdots, 1), \\ D^2 d(x) = \operatorname{diag}\left[\frac{-\rho_1(\bar{x})}{1-d(x)\rho_1(\bar{x})}, \cdots, \frac{-\rho_{n-1}(\bar{x})}{1-d(x)\rho_{n-1}(\bar{x})}, 0\right]. \end{cases} \tag{2.4}$$

Supersolution

Lemma 6. For any $0 < \varepsilon < \underline{b}/2$, let

$$\underline{\xi}_{\varepsilon} = \left(\frac{\underline{b} - 2\varepsilon}{(1+\varepsilon)L_0(1 - C_f^{-1}(1 - C_m))} \right)^{1/(k+1)},$$

where $\underline{b}$, C_m, L_0 and C_f are given by $(\mathbf{b_2})$, and $(\mathbf{f_2})$ respectively. Then for sufficiently small $\delta_\varepsilon > 0$, the following function

$$\overline{u}_\varepsilon(x) = -\varphi(\underline{\xi}_{\varepsilon} M(d(x)))$$

is k-admissible in $\Omega_{\delta_\varepsilon}$ and satisfies

$$S_k(D^2\overline{u}_\varepsilon(x)) \leq b(x)f(-\overline{u}_\varepsilon(x)) \quad \text{in } \Omega_{\delta_\varepsilon},$$

where φ, M and $\Omega_{\delta_\varepsilon}$ are given by $(\mathbf{b_2})$ and $(\mathbf{f_2})$ respectively.

Subsolution

Lemma 7. For any $\varepsilon > 0$, let

$$\overline{\xi}_\varepsilon = \left(\frac{\overline{b} + 2\varepsilon}{(1-\varepsilon) l_0 (1 - C_f^{-1}(1 - C_m))} \right)^{1/(k+1)},$$

where $\overline{b}$, C_m, l_0 and C_f are given by $(\mathbf{b_2})$, and $(\mathbf{f_2})$ respectively. Then for sufficiently small $\delta_\varepsilon > 0$, the following function

$$\underline{u}_\varepsilon(x) = -\varphi(\overline{\xi}_\varepsilon M(d(x)))$$

is k-admissible in $\Omega_{\delta_\varepsilon}$ and satisfies

$$S_k(D^2 \underline{u}_\varepsilon(x)) \geq b(x) f(-\underline{u}_\varepsilon(x)) \quad \text{in } \Omega_{\delta_\varepsilon},$$

where φ, M and $\Omega_{\delta_\varepsilon}$ are given by $(\mathbf{b_2})$ and $(\mathbf{f_2})$ respectively.

Proof of Lemma 6

Step 1. Show that $\overline{u}_\varepsilon$ is a k-admissible function in $\Omega_{\delta_\varepsilon}$ with sufficiently small $\delta_\varepsilon > 0$. That is, for $1 \le j \le k$,

$$S_j(D^2\overline{u}_\varepsilon) > 0 \quad \text{in} \quad \Omega_{\delta_\varepsilon}. \tag{2.5}$$

By direct computation,

$$(\overline{u}_\varepsilon(x))_{\alpha\beta} = (-\varphi(\underline{\xi}_\varepsilon M(d(x))))_{\alpha\beta} =$$
$$-\underline{\xi}_\varepsilon \left[\underline{\xi}_\varepsilon \varphi''\left(\underline{\xi}_\varepsilon M(d(x))\right) m^2(d(x)) + \varphi'\left(\underline{\xi}_\varepsilon M(d(x))\right) m'(d(x))\right] d_\alpha d_\beta$$
$$-\underline{\xi}_\varepsilon\ \varphi'\left(\underline{\xi}_\varepsilon M(d(x))\right) m(d(x))\, d_{\alpha\beta}.$$

Proof of Lemma 6

Using (2.4) and Lemma 5 (i_1), we derive that for $1 \leq j \leq k$,

$$
\begin{aligned}
& S_j(D^2\overline{u}_\varepsilon) \\
&= -\left[\underline{\xi}_\varepsilon^2 \varphi''\left(\underline{\xi}_\varepsilon M(d(x))\right) m^2(d(x)) + \underline{\xi}_\varepsilon \varphi'\left(\underline{\xi}_\varepsilon M(d(x))\right) m'(d(x))\right] \\
&\quad \times \left(\underline{\xi}_\varepsilon \varphi'\left(\underline{\xi}_\varepsilon M(d(x))\right) m(d(x))\right)^{j-1} \sigma_{j-1}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x}} \right. \\
&\quad + \left(\underline{\xi}_\varepsilon \varphi'\left(\underline{\xi}_\varepsilon M(d(x))\right) m(d(x))\right)^{j} \sigma_{j}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) \\
&= \underline{\xi}_\varepsilon^{j+1} m^{j+1}(d(x)) f(\varphi(\underline{\xi}_\varepsilon M(d(x))))\left((k+1)F\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)\right)^{(j-k)} \\
&\quad \times \left[\left(1 - \frac{M(d(x))m'(d(x))}{m^2(d(x))} \frac{\left((k+1)F\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)\right)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)}\right)\right. \\
&\qquad \times \sigma_{j-1}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) \\
&\qquad + \frac{M(d(x))}{m(d(x))} \frac{\left((k+1)F\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)\right)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x)) f\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)} \sigma_j\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}}\right.
\end{aligned}
$$

Proof of Lemma 6

By Lemma 4,

$$\lim_{\substack{x\in\Omega\\ d(x)\to 0}} \frac{M(d(x))}{m(d(x))} = 0 \quad \text{and} \quad \lim_{\substack{x\in\Omega\\ d(x)\to 0}} \frac{M(d(x))m'(d(x))}{m^2(d(x))} = 1 - C_m.$$

By Lemma 5 (i_1) and (i_2),

$$M(d(x)) = \int_0^{\varphi(M(d(x)))} ((k+1)F(\tau))^{-1/(k+1)} d\tau$$

and

$$\lim_{\substack{x\in\Omega\\ d(x)\to 0}} \frac{((k+1)F(\varphi(M(d(x)))))^{k/(k+1)}}{M(d(x))f(\varphi(M(d(x))))} = \frac{1}{C_f}.$$

Proof of Lemma 6

Therefore,

$$1-\frac{M\left(d\left(x\right)\right)m'\left(d\left(x\right)\right)}{m^2\left(d\left(x\right)\right)}\frac{\left(\left(k+1\right)F\left(\varphi\left(\underline{\xi}_{\varepsilon}M\left(d\left(x\right)\right)\right)\right)\right)^{k/(k+1)}}{\underline{\xi}_{\varepsilon}M\left(d\left(x\right)\right)f\left(\varphi\left(\underline{\xi}_{\varepsilon}M\left(d\left(x\right)\right)\right)\right)}>0 \text{ in } \Omega_{\delta}$$

for sufficiently small $\delta_\varepsilon > 0$.

For $1 \leq j \leq k-1$, since Ω being strictly $(k-1)$-convex,

$$\sigma_j\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})},\cdots,\frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right)>0 \quad \text{in} \quad \Omega_{\delta_\varepsilon}$$

as $\delta_\varepsilon > 0$ being sufficiently small. Furthermore,

$$\sigma_k\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})},\cdots,\frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) \quad \text{is bounded in} \quad \Omega_{\delta_\varepsilon}.$$

Therefore (2.5) holds.

Proof of Lemma 6

Step 2. Prove supersolution. By ($\mathbf{b_2}$),

$$(\underline{b}-\varepsilon)m^{k+1}(d(x)) \le b(x) \text{ in } \Omega_{\delta_\varepsilon}$$

for sufficiently small $\delta_\varepsilon > 0$, we see that "supersolution" is an easy consequence of

$$S_k(D^2\overline{u}_\varepsilon(x)) \le (\underline{b}-\varepsilon)\, m^{k+1}(d(x))\, f(-\overline{u}_\varepsilon(x)) \text{ in } \Omega_{\delta_\varepsilon}. \quad (2.6)$$

By above calculations,

$$S_k(D^2\overline{u}_\varepsilon(x)) - (\underline{b}-\varepsilon)\, m^{k+1}(d(x))\, f(-\overline{u}_\varepsilon(x))$$
$$= \underline{\xi}_\varepsilon^{k+1} m^{k+1}(d(x)) f(\varphi(\underline{\xi}_\varepsilon M(d(x))))$$
$$\times \Bigg[\Bigg(1 - \frac{M(d(x))m'(d(x))}{m^2(d(x))} \frac{\big((k+1)F\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)\big)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x)) f\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)} \Bigg)$$
$$\times \sigma_{k-1}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})} \right)$$
$$+ \frac{M(d(x))}{m(d(x))} \frac{\big((k+1)F\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)\big)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x)) f\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)} \sigma_k \left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-}}\right.$$

Proof of Lemma 6

We only need to prove

$$
\begin{aligned}
&\underline{\xi}_\varepsilon^{k+1}\Bigg[\left(1-\frac{M(d(x))m'(d(x))}{m^2(d(x))}\frac{\Big((k+1)F\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)\Big)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x))f\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)}\right)\\
&\quad\times\sigma_{k-1}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})},\cdots,\frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right)\\
&\quad+\frac{M(d(x))}{m(d(x))}\frac{\Big((k+1)F\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)\Big)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x))f\big(\varphi\big(\underline{\xi}_\varepsilon M(d(x))\big)\big)}\sigma_k\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})},\cdots,\frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-}}\right.\\
&-(\underline{b}-\varepsilon)\le 0\quad\text{in}\quad\Omega_{\delta_\varepsilon}
\end{aligned}\tag{2.7}
$$

for sufficiently small $\delta_\varepsilon > 0$.

Proof of Lemma 6

By definition of $\underline{\xi}_\varepsilon$,

$$\underline{\xi}_\varepsilon^{k+1}\left[(1+\varepsilon)L_0(1-C_f^{-1}(1-C_m))\right]-(\underline{b}-\varepsilon)=-\varepsilon.$$

It follows that

$$\underline{\xi}_\varepsilon^{k+1}\Bigg[(1+\varepsilon)L_0\left(1-\frac{M(d(x))m'(d(x))}{m^2(d(x))}\frac{\left((k+1)F\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)\right)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x))f\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)}\right)$$
$$+\frac{M(d(x))}{m(d(x))}\frac{\left((k+1)F\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)\right)^{k/(k+1)}}{\underline{\xi}_\varepsilon M(d(x))f\left(\varphi\left(\underline{\xi}_\varepsilon M(d(x))\right)\right)}\sigma_k\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})},\cdots,\frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-}}\right.$$
$$-(\underline{b}-\varepsilon)\le 0 \quad \text{in} \quad \Omega_{\delta_\varepsilon}$$

for sufficiently small $\delta_\varepsilon>0$.

Proof of Lemma 6

Furthermore,

$$\sigma_{k-1}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})},\cdots,\frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right)\leq(1+\varepsilon)L_0 \quad \text{in} \quad \Omega_{\delta_\varepsilon}.$$

Combining above inequalities, we obtain (2.7). Therefore we have (2.6).

We prove Lemma 7 by a similar way.

Proof of Theorem 2

Step (i). We prove the first inequality of (2.3).

Let $v \in C^2(\overline{\Omega})$ to be the k-admissible solution of $S_k(D^2 v) = 1$ in Ω with $v = 0$ on $\partial\Omega$.

Since $\Delta v > 0$ in Ω, we have $v \leq 0$ in Ω. There exists a negative constant c such that

$$cd(x) \leq v(x) \text{ on } \overline{\Omega}.$$

Then we have, for sufficiently large T,

$$u + Tv \leq \overline{u}_\varepsilon \text{ on } \Lambda = \{x \in \Omega : d(x) = \delta_\varepsilon\}$$

and

$$u + Tv = \overline{u}_\varepsilon = 0 \text{ on } \partial\Omega.$$

Proof of Theorem 2

It is clear that

$$D^2u + TD^2v \in S(\Gamma_k) \ \text{ in } \Omega.$$

Using the concavity of $S_k^{1/k}$ on $S(\Gamma_k)$,

$$S_k^{1/k}\left(\frac{1}{2}D^2(u+Tv)\right) \geq \frac{1}{2}S_k^{1/k}(D^2u)+\frac{1}{2}S_k^{1/k}(TD^2v) \geq \frac{1}{2}S_k^{1/k}(D^2u) \ \text{ in}$$

Therefore,

$$S_k(D^2(u+Tv)) \geq S_k(D^2u) = b(x)f(-u) \geq b(x)f(-(u+Tv)) \ \text{ in } \Omega.$$

By Lemma 3,

$$u + Tv \leq \overline{u}_\varepsilon = -\varphi(\underline{\xi}_\varepsilon M(d(x))) \ \text{ in } \Omega_{\delta_\varepsilon}.$$

Proof of Theorem 2

Divide both sides by $-\varphi(\underline{\xi}_\varepsilon M(d(x)))$ and then

$$1 - \frac{cTd(x)}{-\varphi(\underline{\xi}_\varepsilon M(d(x)))} \le \frac{u}{-\varphi(\underline{\xi}_\varepsilon M(d(x)))} \quad \text{in } \Omega_{\delta_\varepsilon}.$$

Since, by Lemma 5 (i_3),

$$\lim_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{d(x)}{\varphi(\underline{\xi}_\varepsilon M(d(x)))} = 0,$$

we obtain

$$1 \le \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\underline{\xi}_\varepsilon M(d(x)))}.$$

Let $\varepsilon \to 0$ and then we conclude

$$1 \le \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\underline{\xi} M(d(x)))}.$$

Proof of Theorem 2

Step (ii). We turn to prove the second inequality of (2.3).

For sufficiently large $T > 0$, $\underline{u}_\varepsilon + Tv \leq u$ in $\Omega_{\delta_\varepsilon}$. That is,

$$-\varphi(\bar{\xi}_\varepsilon M(d(x))) + Tv \leq u \quad \text{in } \Omega_{\delta_\varepsilon}.$$

Divide both sides by $-\varphi(\bar{\xi}_\varepsilon M(d(x)))$ and then

$$\frac{u}{-\varphi(\bar{\xi}_\varepsilon M(d(x)))} \leq 1 + \frac{cTd(x)}{-\varphi(\bar{\xi}_\varepsilon M(d(x)))} \quad \text{in } \Omega_{\delta_\varepsilon}.$$

It follows that

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\bar{\xi}_\varepsilon M(d(x)))} \leq 1.$$

Let $\varepsilon \to 0$ and we obtain

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{-\varphi(\bar{\xi} M(d(x)))} \leq 1.$$

Recall (1.2)

$$\begin{cases} S_k(D^2u) = \sigma_k(\lambda) = b(x)f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Assumptions on f and b.

($\mathbf{f_1}$) $f \in C^1(0,\infty)$, $f(s) > 0$, and is nondecreasing in $(0,\infty)$;
($\mathbf{f_2}$) The function

$$\Phi(s) = \int_s^\infty \frac{d\tau}{H(\tau)}$$

is well defined for any $s > 0$, where $H(\tau) = ((k+1)F(\tau))^{1/(k+1)}$ and $F(\tau) = \int_0^\tau f(s)ds$. For convenience, we define by φ the inverse of Φ, i.e., φ satisfies

$$\int_{\varphi(t)}^\infty \frac{d\tau}{H(\tau)} = t \quad \forall\ 0 < t < \alpha,$$

($\mathbf{b_1}$) $b \in C^{1,1}(\overline{\Omega})$ is positive in Ω.

Existence of large solutions

Theorem 3: Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial\Omega \in C^{3,1}$. Suppose that f satisfies $(\mathbf{f_1})$ and $(\mathbf{f_2})$, and that b satisfies $(\mathbf{b_1})$. Then problem (1.1) admits a viscosity solution $u \in C(\Omega)$.

Some related results:

P. Salani, Boundary blow-up problems for Hessian equations, Manus. Math. 96 (1998) 281-294.

A. Colesanti, P. Salani, E. Francini, Convexity and asymptotic estimates for large solutions of Hessian equations, Differential Integral Equations 13 (2000) 1459-1472.

Y. Huang, Boundary asymptotical behavior of large solutions to Hessian equations, Pacific J. Math. 244 (2010) 85-98.

H.Y. Jian, Hessian equations with infinite Dirichlet boundary value, Indiana Univ. Math. J. 55 (2006) 1045-1062.

Proof of Theorem 3

We use the method of proving Theorem 1.
Let w $(w < 0)$ is the admissible solution of

$$\begin{cases} S_k(D^2 w) = b(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $w \in C^{3,\beta}(\overline{\Omega})$ with $0 < \beta < 1$.
$(\mathbf{f_2})$ implies that

$$\Psi(s) = \int_s^\infty \frac{1}{f(\tau)^{1/k}} d\tau$$

is well defined. Let ψ be the inverse of Ψ, i.e., ψ satisfies

$$t = \int_{\psi(t)}^\infty \frac{1}{f(\tau)^{1/k}} d\tau.$$

Proof of Theorem 3

Define

$$\underline{h}(x) = \psi(-w(x)) \quad \text{in } \Omega,$$

and $j = 1, 2, \cdots$,

$$\Omega_j = \{x \in \Omega : \underline{h}(x) < j\}.$$

Note that $\underline{h} = \infty$ on $\partial\Omega$.

Consider

$$\begin{cases} S_k(D^2u) = b(x)f(u) & \text{in } \Omega_j, \\ u = j & \text{on } \partial\Omega_j. \end{cases}$$

Since $\underline{h}$ is a k-admissible subsolution, we have, by Lemma 2, it has k-admissible solution u_j.

It is clear

$$u_j \leq u_{j+1} \quad \text{in } \Omega_j.$$

Construction supersolution

Lemma 8. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial\Omega \in C^{3,1}$. Suppose that f satisfies $(\mathbf{f_1})$ and $(\mathbf{f_2})$, and that $b \in C^{1,1}(\overline{\Omega})$ is positive. Then there exists a $\overline{h} \in C^2(\Omega)$, $\overline{h}(x) \to \infty$ as $d(x) \to 0$, such that for any k-admissible function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$S_k(D^2u) = b(x)f(u) \quad in\ \Omega,$$

we have

$$u \leq \overline{h} \quad in\ \Omega.$$

A. Mohammed, On the existence of solutions to the Monge-Ampère equation with infinite boundary values, Proc. Amer. Math. Soc. 135 (2007) 141-149.

Proof of Theorem 3

For each $x_0 \in \Omega$, let $B_R(x_0) \subset\subset \Omega$ with $0 < R < d(x_0)$. Then there exists a positive integer j_0 so that $B_R(x_0) \subset \Omega_{j_0}$. Since $b \in C^{1,1}(\overline{B_R(x_0)})$ is positive, by Lemma 8, there exists a $\overline{h}_R \in C^2(B_R(x_0))$, $\overline{h}_R(x) \to \infty$ as $\mathrm{dist}(x, \partial B_R(x_0)) \to 0$, such that for all $j \geq j_0$,

$$u_j \leq \overline{h}_R \quad \text{in } B_R(x_0).$$

It follows that

$$\lim_{j\to\infty} u_j(x_0) = u(x_0).$$

We also have

$$\|u_j\|_{C^\beta(\overline{B_r(x_0)})} \leq C.$$

Then $u \in C(\Omega)$.

Moreover, $u = \infty$ on $\partial\Omega$ since $\underline{h} \leq u$ in Ω and $\underline{h} = \infty$ on $\partial\Omega$. That is, u is the viscosity solution.

Further assumptions on f and b

($\mathbf{f_3}$) There exists $C_f > 0$ such that

$$\lim_{s\to\infty} H'(s) \int_s^\infty \frac{d\tau}{H(\tau)} = C_f,$$

where $H(\tau)$ is defined in ($\mathbf{f_2}$).

($\mathbf{b_2}$) There exist a positive and nondecreasing function $m(t) \in C^1(0, \delta_0)$ (for some $\delta_0 > 0$), and two positive constants $\overline{b}$ and $\underline{b}$ such that

$$\underline{b} = \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{b(x)}{m^{k+1}(d(x))} \le \limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{b(x)}{m^{k+1}(d(x))} = \overline{b}.$$

Moreover, there exists $C_m \in [0, \infty)$ such that

$$\lim_{t\to 0^+} \left(\frac{M(t)}{m(t)}\right)' = C_m,$$

Theorem 4

Theorem 4: Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and strictly $(k-1)$-convex domain with $\partial\Omega \in C^{3,1}$. Suppose that f satisfies $(\mathbf{f_1})$, $(\mathbf{f_2})$ and $(\mathbf{f_3})$, and that b satisfies $(\mathbf{b_1})$ and $(\mathbf{b_2})$. If

$$C_f > 1 - C_m, \tag{3.1}$$

then every viscosity solution u of (1.1) satisfies

$$1 \leq \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{\varphi(\overline{\xi} M(d(x)))} \quad \text{and} \quad \limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{\varphi(\underline{\xi} M(d(x)))} \leq 1, \tag{3.2}$$

where

$$\underline{\xi} = \left(\frac{\underline{b}}{L_0(1 - C_f^{-1}(1 - C_m))} \right)^{\frac{1}{k+1}}, \ \overline{\xi} = \left(\frac{\overline{b}}{l_0(1 - C_f^{-1}(1 - C_m))} \right)^{\frac{1}{k+1}}.$$

Remark

If b satisfies $(\mathbf{b_1})$ and $(\mathbf{b_2})$, and f satisfies $(\mathbf{f_1})$, $(\mathbf{f_2})$ and $(\mathbf{f_3})$, then $0 \leq C_m \leq 1$ and $C_f \geq 1$.

c.f. Z.J. Zhang, Boundary behavior of large solutions to the Monge-Ampère equations with weights, J. Differential Equations 259 (2015) 2080-2100.

Hence (3.1) holds if $C_f > 1$, or $C_f = 1$ and $C_m > 0$.

Example 1

Set $b = d(x)^{\alpha(k+1)}$, $0 \le \alpha < +\infty$, near $\partial\Omega$ and $f(s) = s^\gamma$, $\gamma > k$. Choose $m(t) = t^\alpha$ and then

$$M(t) = \frac{t^{\alpha+1}}{\alpha+1} \quad \text{and} \quad C_m = \frac{1}{\alpha+1}.$$

We also have $C_f = \frac{\gamma+1}{\gamma-k} > 1$. We need $\gamma > k$. Then

$$\varphi(t) = \left(\frac{(k+1)^k(\gamma+1)}{(\gamma-k)^{k+1}}\right)^{1/(\gamma-k)} t^{-(k+1)/(\gamma-k)}.$$

Example 1

By Theorem 4,

$$\underline{\xi} = \left(\frac{(\alpha+1)(\gamma+1)}{L_0(\gamma+1+\alpha k+\alpha)} \right)^{1/(k+1)} \quad \text{and} \quad \overline{\xi} = \left(\frac{(\alpha+1)(\gamma+1)}{l_0(\gamma+1+\alpha k+\alpha)} \right)^{1/(k+1)}$$

$$1 \le \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{\left(\frac{l_0(\gamma+\alpha k+\alpha+1)(k+1)^k(\alpha+1)^k}{(\gamma-k)^{k+1}}\right)^{1/(\gamma-k)} d(x)^{-(k+1)(\alpha+1)/(\gamma-k)}}$$

and

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{\left(\frac{L_0(\gamma+\alpha k+\alpha+1)(k+1)^k(\alpha+1)^k}{(\gamma-k)^{k+1}}\right)^{1/(\gamma-k)} d(x)^{-(k+1)(\alpha+1)/(\gamma-k)}} \le 1.$$

Example 2

Set $b = d(x)^{\alpha(k+1)}$, $0 \leq \alpha < +\infty$, near $\partial\Omega$ and $F(s) = e^s$, $s > S_0$ for some large S_0. Choose $m(t) = t^\alpha$. We have $C_f = 1$,

$$\varphi(t) = k \ln(k+1) - (k+1) \ln t,$$

$$\underline{\xi} = \left(\frac{\alpha+1}{L_0}\right)^{1/(k+1)} \quad \text{and} \quad \overline{\xi} = \left(\frac{\alpha+1}{l_0}\right)^{1/(k+1)}.$$

By Theorem 4,

$$1 \leq \liminf_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{k \ln(k+1)(\alpha+1) + \ln l_0 - (k+1)(\alpha+1) \ln d(x)}$$

and

$$\limsup_{\substack{x\in\Omega \\ d(x)\to 0}} \frac{u(x)}{k \ln(k+1)(\alpha+1) + \ln L_0 - (k+1)(\alpha+1) \ln d(x)} \leq 1.$$

Some related results

For $k = 1$, Laplace equation.

L. Bieberbach, $\Delta u = e^u$ und die automorphen Funktionen, Math. Ann. 77 (1916) 173-212.

C. Bandle, M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. Anal. Math. 58 (1992) 9-24.

C. Bandle, M. Marcus, Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (1995) 155-171.

F. Cîrstea, V. Rădulescu, Asymptotics for the blow-up boundary solution of the logistic equation with absorption, C. R. Math. Acad. Sci. Paris 336 (2003) 231-236.

J. García-Melián, Nondegeneracy and uniqueness for boundary blow-up elliptic problems, J. Differential Equations 223 (2006) 208-227.

Some related results

For $k = n$, the Monge-Ampère equation.

F.C. Cîrstea, C. Trombetti, On the Monge-Ampère equation with boundary blow-up: existence, uniqueness and asymptotics, Calc. Var. Partial Differential Equations 31 (2008) 167-186.

B. Guan, H.Y. Jian, The Monge-Ampère equation with infinite boundary value, Pacific J. Math. 216 (2004) 77-94.

A.C. Lazer, P.J. McKenna, On singular boundary value problems for the Monge-Ampère operator, J. Math. Anal. Appl. 197 (1996) 341-362.

C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal or projective transformations, in: Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers), Academic Press, New York, 1974: 245-272.

A. Mohammed, On the existence of solutions to the Monge-Ampère equation with infinite boundary values, Proc. Amer. Math. Soc. 135 (2007) 141-149.

Some related results

For general k-Hessian equation.

P. Salani, Boundary blow-up problems for Hessian equations, Manus. Math. 96 (1998) 281-294.

Y. Huang, Boundary asymptotical behavior of large solutions to Hessian equations, Pacific J. Math. 244 (2010) 85-98.

Two Lemmas

Lemma 9. Let m and M be the functions given by $(\mathbf{b_2})$. Then

$$M(0)=\lim_{t\to 0^+}M(t)=0,$$

$$\lim_{t\to 0^+}\frac{M(t)}{m(t)}=0,$$

and

$$\lim_{t\to 0^+}\frac{M(t)m'(t)}{m^2(t)}=1-C_m.$$

Two Lemmas

Lemma 10. Assume that f satisfies $(\mathbf{f_1})$, $(\mathbf{f_2})$ and $(\mathbf{f_3})$. Then we have

(i_1) $\varphi(t) > 0$, $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = +\infty$,

$$\varphi'(t) = -\left((k+1)F(\varphi(t))\right)^{1/(k+1)},$$

$$\textit{and } \varphi''(t) = \left((k+1)F(\varphi(t))\right)^{(1-k)/(k+1)} f(\varphi(t));$$

(i_2) $\lim_{t\to 0^+} \frac{-\varphi'(t)}{t\varphi''(t)} = \lim_{t\to 0^+} \frac{((k+1)F(\varphi(t)))^{k/(k+1)}}{tf(\varphi(t))} = \frac{1}{C_f}$;

(i_3) $\varphi \in NRVZ_{1-C_f}$, *i.e.*, for each $\xi > 0$,

$$\lim_{t\to 0^+} \frac{\varphi(\xi t)}{\varphi(t)} = \xi^{1-C_f}.$$

Proof of Theorem 4

For any $\varepsilon > 0$, we choose $\delta_\varepsilon > 0$ small enough such that

($\mathbf{a_1}$) $m(t)$ satisfies ($\mathbf{b_2}$) for $0 < t < \delta_\varepsilon$;

($\mathbf{a_2}$) $d(x) \in C^2(\Omega_{2\delta_\varepsilon})$;

($\mathbf{a_3}$) $(\underline{b} - \varepsilon)m^{k+1}(d(x)) \leq b(x) \leq (\overline{b} + \varepsilon)m^{k+1}(d(x))$ in $\Omega_{2\delta_\varepsilon}$;

($\mathbf{a_4}$) For any $0 \leq j \leq k-1$,
$\sigma_j\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) > 0$ in $\Omega_{2\delta_\varepsilon}$. Recall that $\rho_i(\overline{x})$ $(i = 1, 2, \cdots, n-1)$ denote the principal curvatures of $\partial\Omega$ at $\overline{x}$, where $\overline{x} \in \partial\Omega$ satisfies $d(x) = |x - \overline{x}|$;

($\mathbf{a_5}$)
$(1-\varepsilon)l_0 \leq \sigma_{k-1}\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right) \leq (1+\varepsilon)L_0$ in $\Omega_{2\delta_\varepsilon}$;

($\mathbf{a_6}$) $\sigma_k\left(\frac{\rho_1(\overline{x})}{1-d(x)\rho_1(\overline{x})}, \cdots, \frac{\rho_{n-1}(\overline{x})}{1-d(x)\rho_{n-1}(\overline{x})}\right)$ is bounded in $\Omega_{2\delta_\varepsilon}$.

Proof of Theorem 4

Fix $0 < \varepsilon < \underline{b}/2$ and we choose

$$\underline{\xi}_\varepsilon = \left(\frac{\underline{b} - 2\varepsilon}{(1+\varepsilon)L_0(1 - C_f^{-1}(1 - C_m))} \right)^{1/(k+1)},$$

and

$$\overline{\xi}_\varepsilon = \left(\frac{\overline{b} + 2\varepsilon}{(1-\varepsilon)l_0(1 - C_f^{-1}(1 - C_m))} \right)^{1/(k+1)},$$

Proof of Theorem 4

Choose $0 < \sigma < \delta_\varepsilon$.
Define

$$d_1(x) = d(x) - \sigma, \quad d_2(x) = d(x) + \sigma$$

and

$$\begin{cases} \overline{u}_\varepsilon(x) = \varphi(\underline{\xi}_\varepsilon M(d_1(x))) \text{ in } \Omega_{2\delta_\varepsilon} \setminus \overline{\Omega}_\sigma, \\ \underline{u}_\varepsilon(x) = \varphi(\overline{\xi}_\varepsilon M(d_2(x))) \text{ in } \Omega_{2\delta_\varepsilon - \sigma}. \end{cases}$$

Proof of Theorem 4

Step 1. $\overline{u}_\varepsilon$ is k-admissible and

$$S_k(D^2\overline{u}_\varepsilon(x)) \leq b(x)f(\overline{u}_\varepsilon(x)) \quad \text{in } \Omega_{2\delta_\varepsilon} \setminus \overline{\Omega}_\sigma$$

as δ_ε being sufficiently small.

Step 2. $\underline{u}_\varepsilon$ is k-admissible and

$$S_k(D^2\underline{u}_\varepsilon(x)) \geq b(x)f(\underline{u}_\varepsilon(x)) \quad \text{in } \Omega_{2\delta_\varepsilon - \sigma}$$

as δ_ε being sufficiently small.

Proof of Theorem 4.

Step 3. Let $T > 0$ sufficiently large and $0 < \sigma < \delta_\varepsilon$. Then

$$u \le \overline{u}_\varepsilon + T \quad \text{on } \Lambda_1 = \{x \in \Omega : d(x) = 2\delta_\varepsilon\}$$

and

$$\underline{u}_\varepsilon \le u + T \quad \text{on } \Lambda_2 = \{x \in \Omega : d(x) = 2\delta_\varepsilon - \sigma\}.$$

We observe that

$$u \le \overline{u}_\varepsilon + T = \infty \quad \text{on } \Lambda_3 = \{x \in \Omega : d(x) = \sigma\}$$

and

$$\underline{u}_\varepsilon \le u + T = \infty \quad \text{on } \partial\Omega.$$

By Lemma 3,

$$\underline{u}_\varepsilon \le u + T \quad \text{in } \Omega_{2\delta_\varepsilon - \sigma}.$$

and

$$\underline{u}_\varepsilon \le u + T \quad \text{in } \Omega_{2\delta_\varepsilon - \sigma}.$$

Thank you!