Geometric similarity invariants of geometric operators

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outline

1. Cowen-Douglas Operators

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Cowen-Douglas Operators

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$.
For an open connected subset Ω of the complex plane $\mathbb{C}$, and $n \in \mathbb{N}$, Cowen and Douglas introduced the class of operators $B_n(\Omega)$ in their Acta paper [1].

Definition ($B_n(\Omega)$)

An operator $T \in B_n(\Omega)$ if for each $w \in \Omega$, is an eigenvalue of the operator T of constant multiplicity n, these eigenvectors span the Hilbert space $\mathcal{H}$ and the operator $T - w,\ w \in \Omega$, is surjective.

It was showed that the map $w \to \ker(T - w)$ is holomorphic and $\pi : E_T \to \Omega$, where

$$E_T(w) = \{\ker(T - w) : w \in \Omega\}, \pi(\ker(T - w)) = w$$

defines a Hermitian holomorphic vector bundle on Ω.

Cowen-Douglas Operators

First of all, we need introduce some complex geometry notations:
Let $\xi(\Omega)$ be the algebra consist of the C^∞ functions and $\xi^p(\Omega)$ denote the p-differential form of C^∞ functions. Thus we have

$$\xi^0(\Omega) = \xi(\Omega), \xi^1(\Omega) = \{fdz + gd\bar{z} : f, g \in \xi(\Omega)\},$$

$$\xi^2(\Omega) = \{fdzd\bar{z}, f \in \xi(\Omega)\}$$

For any vector bundle E which has C^∞ differential structure, let $\xi^p(\Omega, E)$ denotes p-differential forms with the coefficients in E. Then each element in $\xi^0(\Omega, E)$ is one of sections of E.

Cowen-Douglas Operators

Definition (Connection and Curvature)

The connection D can be regarded as a differential operator which maps $\xi^0(\Omega, E)$ *to* $\xi^1(\Omega, E)$. *Let* $\sigma \in E(w)$, *and* $h = ((\langle \sigma_j, \sigma_i \rangle))_{n \times n}$. *Then the canonical connection D which keeping the metric and satisfying the following equality:*

$$D(\sum_{i=1}^{n} f_i \sigma_i) = \sum_{i=1}^{n} df_i \otimes \sigma_i + \sum_{i=1}^{n} \sum_{j=1}^{n} f_i \theta_{j,i} \sigma_j$$

where $\theta = h^{-1} \partial h$. *And*

$$D^2 = d\theta + \theta \wedge \theta = \bar{\partial}(h^{-1} \partial h)$$

then $-\bar{\partial}(h^{-1} \partial h)$ *is called as the curvature of E denoted by* K_E

Definition (Second fundamental form)

Let $T \in B_2(\Omega)$, and $\sigma_1(w), \sigma_2(w) \in Ker(T - w)$. Applying the Schmidt orthogonal progress to σ_1, σ_2, then we have e_1, e_2. Suppose

$$De_1 = D^{1,0}e_1 + D^{0,1}e_2 = \theta_{11}e_1 + \theta_{21}e_2$$

and $De_2 = \theta_{12}e_1 + \theta_{22}e_2$, then $\theta_{12} = \langle De_2, e_1 \rangle$ is called as the second fundamental form of E_T

Questions

Cowen-Douglas' Unitary Classification Theorem

For any $T \in B_n(\Omega)$, when $n = 1$, the curvature is the completely unitary invariant. When $n > 1$, then the curvature and it's covariant partial derivatives are the completely unitary invariants

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Question 1(Similarity of Cowen-Douglas Operators)

For the similarity of Cowen-Douglas operators $A, B \in B_1(\mathbb{D})$, whether $A \sim_s B$ if and only if

$$\lim_{w \to \partial \mathbb{D}} \frac{K_A(w)}{K_B(w)} = 1.$$

Or can we use some geometric invariants involving curvature to describe the similarity of Cowen-Douglas operators?

D.N. Clark and G. Misra's result

D. N. Clark and G. Misra gave a counter example of this conjecture. Let S_0 be the backward unilateral shift operator and T be a weighted (backward) shift operator with sequence $\alpha_n = \frac{(\sum\limits_{j=1}^{n} 1/j)^{1/2}}{(\sum\limits_{j=1}^{n+1} 1/j)^{1/2}}$ and

$$\begin{aligned} \frac{K_T}{K_{S_0}} &= 1 + [1 - ln(1 - |w|^2)]^{-1} \\ &\quad - |w|^2[1 - ln(1 - |w|^2)]^{-2} \end{aligned}$$

Then $\frac{K_T}{K_{S_0}} \to 1$, when $|w|$ goes to 1. However, T and S_0 are not similar.

D.N. Clark and G. Misra's result

Theorem [D.N.Clark and G.Mirsa]Michigan Math. J. 1983

Let S denote a backward weighted shift operator with weight sequence $\alpha_n = [\frac{(n+1)}{(n+2)}]^{\alpha/2}$ and T is a backward weighted shift operator with $||T|| \leq 1$. Set α_w to be the ratio of the normalized sections of E_S and E_T. Then

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(i) T is similar to S if and only if α_w is bounded and bounded from 0.

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(i) T is similar to S if and only if α_w is bounded and bounded from 0.

(ii) T is similar to S with $T = XSX^{-1}$, $X = U + K$ where U is unitary and K is compact if and only if α_w tends to a non-zero limit when $|w| \to 1$.

Kehe Zhu's result

In [3], K. Zhu introduced the spanning holomorphic cross-section for Cowen-Douglas operators. Let $T \in B_n(\Omega)$. A holomorphic section of vector bundle E_T is a holomorphic function $\gamma : \Omega \to \mathcal{H}$ such that for each $w \in \Omega$, the vector $\gamma(w)$ belongs to the fibre of E_T over w. We say γ is a spanning holomorphic section for E_T if $\overline{\text{Span}}\, \{\gamma(w) : w \in \Omega\} = \mathcal{H}$.

Theorem [K. Zhu] Illinois J. Math. 2000

For any Cowen-Douglas operator $T \in B_n(\Omega)$, E_T has a spanning holomorphic cross-section. Suppose T and $\widetilde{T}$ belongs to $\mathcal{B}_n(\Omega)$, then T and $\widetilde{T}$ are unitarily equivalent (or similarity equivalent) if and only if there exist spanning holomorphic cross-sections γ_T and $\gamma_{\widetilde{T}}$ for E_T and E_S, respectively, such that $\gamma_T \sim_u \gamma_{\widetilde{T}}$ (or $\gamma_T \sim_s \gamma_{\widetilde{T}}$).

R.G.Douglas, S.Treil and H. Kwon's result

Theorem [R.G. Douglas, H. Kwon and S.Treil] J. Lond. Math. Soc. 2013

For $T \in B_m(\mathbb{D})$ that is an n-hypercontraction, let $P : \mathbb{D} \to \mathcal{L}(\mathcal{H})$ denote the function whose values are orthogonal projections onto $\ker(T - w)$.

Then T is similar to $\bigoplus^{m} S_n^*$ if and only if there exists a bounded subharmonic function ψ defined on $\mathbb{D}$ such that

$$||\partial P(w)||_2^2 - \frac{mn}{(1 - |w|^2)^2} = \Delta\psi(w),$$

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Remark[Y. Hou, K. Ji and H. Kwon] Studia Math. 2017

The Hilbert-Schmidt norm $||\partial P(w)||_2^2$ is pointed out to be $-traceK_T$.

Homogeneous operators

In 1984, G. Misra defined a class of homogeneous Cowen-Douglas operators as the following: an operator T is said to be homogeneous if $\phi(T)$ is unitarily equivalent to T for each Möbius transformation ϕ, and he proved the following theorem:

Theorem[G. Misra]Proc. Amer. Math. Soc.1984

Let $T \in B_1(\mathbb{D})$ is a homogenous operator, then T is unitarily equivalent to the adjoint of multiplication operator M_z on the analytic functional space $\mathcal{H}_{\mathcal{K}}$, where $\mathcal{K}(z,w) = \frac{1}{(1-z\bar{w})^{\lambda}}$, for some $\lambda > -1$.

An operator T is said to be weakly homogeneous if $\phi(T)$ is similarity equivalent to T for each Möbius transformation ϕ. A natural question is what is the set of all of the weakly homogenous operator at least for Cowen-Douglas class?

New class of Cowen-Douglas operators

Inspire of the structure of homogenous Cowen-Douglas operators, we introduced the following new class of operators:

$FB_n(\Omega)$

We let $\mathcal{F}B_n(\Omega)$ be the set of all bounded linear operators T defined on some complex separable Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n-1}$, which are of the form

$$T = \begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1} \\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1} \\ 0 & \cdots & \cdots & 0 & T_{n-1} \end{pmatrix},$$

where the operator $T_i \in B(\mathcal{H}_i)$ is assumed to be in $B_1(\Omega)$ and $T_iS_{i,i+1} = S_{i,i+1}T_{i+1}$, $0 \leq i \leq n-2$.

Second fundamental form in the case of $FB_2(\Omega)$

The 2×2 block $\begin{pmatrix} T_i & S_{ii+1} \\ 0 & T_{i+1} \end{pmatrix}$ in the decomposition of the operator T in $\mathcal{F}B_2(\mathbb{D})$ because of the intertwining property . Hence the corresponding second fundamental form $\theta_{i,i+1}(T)$ is given by the formula

$$\theta_{i,i+1}(T)(z) = \frac{\mathcal{K}_{T_i}(z)\, d\bar{z}}{\left(\frac{\|S_{i,i+1}(t_{i+1}(z))\|^2}{\|t_{i+1}(z)\|^2} - \mathcal{K}_{T_i}(z)\right)^{1/2}}. \tag{2.1}$$

Remark

For any $T, \widetilde{T} \in FB_n(\Omega)$, when $K_{T_i} = K_{\widetilde{T}_i}$, then

$$\theta_{i,i+1}(T)(z) = \theta_{i,i+1}(\widetilde{T})(z) \Leftrightarrow \frac{\|S_{i,i+1}(t_{i+1}(z))\|}{\|t_{i+1}(z)\|} = \frac{\|\widetilde{S}_{i,i+1}(\widetilde{t}_{i+1}(z))\|}{\|\widetilde{t}_{i+1}(z)\|}$$

So we also use $\frac{\|S_{i,i+1}(t_{i+1}(z))\|}{\|t_{i+1}(z)\|}$ as the second fundamental form $\theta_{i,i+1}(T)$.

Unitarily equivalence of operators in $FB_n(\Omega)$

For the unitarily classification problem of Cowen-Douglas operators, we have the following result:

Theorem 1[Jiang,Ji,Dinesh and Misra]JFA, 2017

Let $T, \widetilde{T} \in FB_n(\Omega)$.

$$T \sim_u \widetilde{T} \Leftrightarrow \begin{cases} K_{T_i} = K_{\widetilde{T}_i} \\ \theta_{i,i+1}(T) = \theta_{i,i+1}(\widetilde{T}) \\ \frac{\langle S_{i,j}(t_j), t_i \rangle}{\|t_i\|^2} = \frac{\langle \widetilde{S}_{i,j}(\widetilde{t}_j), \widetilde{t}_i \rangle}{\|\widetilde{t}_i\|^2} \end{cases}$$

Note that numbers of unitarily invariants of common case are n^2. But together with the curvature and the second fundamental form, we find a set of $n(n-1)/2 + 1$ invariants, which are less and easy to compute.

Similarity of operators in $FB_n(\Omega)$

Definition

Let $T \in FB_n(\Omega)$. The operator T is called as quasi-homogeneous operator , i.e. $T \in \mathcal{QB}_n(\Omega)$, if T_i is homogenous operator and

$$S_{i,j}(t_j) \in \bigvee \{t_i^{(k)}, k \leq j-i-1\}.$$

For the similarity classification of Cowen-Douglas operators, we have the following result:

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For the similarity classification of Cowen-Douglas operators, we have the following result:

Theorem 2[Jiang, Ji and Misra] JFA,2017

Let $T, S \in \mathcal{QB}_n(\Omega)$, then we have

$$\begin{cases} K_{T_{i,i}} = K_{\widetilde{T}_{i,i}} \\ \theta_{i,i+1}(T) = \theta_{i,i+1}(\widetilde{T}) \end{cases} \Longrightarrow T \sim_s \widetilde{T} \text{ if and only if } T = \widetilde{T}$$

New progresses

We would like to introduce the following a class of geometric operators denoted by $\mathcal{CFB}_n(\Omega)$.

Definition ($\mathcal{C}FB_n(\Omega)$)

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(2) *$diag\{T\} := T_{1,1} \dotplus T_{2,2} \dotplus \cdots \dotplus T_{n,n} \in \{T\}'$, where $\{T\}'$ denotes the commutant of T;*

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(3) *each entry $T_{i,j} = \phi_{i,j} T_{i,i+1} T_{i+1,i+2} \cdots T_{j-1,j}$, where $\phi_{i,j} \in \{T_{i,i}\}'$;*

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(4) *T is a strongly irreducible operator, i.e. there are no nontrivial idempotents in $\{T\}'$.*

New progresses

Definition (Similarity invariant set)

Let $\mathcal{F} = \{A_\alpha \in \mathcal{B}(\mathcal{H}), \alpha \in \Lambda\}$. *We call* $\mathcal{F}$ *is a similarity invariant set, if for any invertible operator* $X \in \mathcal{B}(\mathcal{H})$,

$$X\mathcal{F}X^{-1} = \{XA_\alpha X^{-1} : A_\alpha \in \mathcal{F}\} = \mathcal{F}.$$

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$\mathcal{CFB}_n(\Omega)$ *is a similarity invariant set*

Remark

The set of homogenous operators in Cowen-Douglas class is not a similarity invariant set.

New progresses

Similarity orbit Theorem(Special case, Apostol, Fialkow, Herrero, and Voiculescu)

Let T and $S \in B_n(\Omega)$, and spectral pictures of T and S be the same. Then there exist two sequences of invertible operators $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} X_n A X_n^{-1} = B, \lim_{n\to\infty} Y_n B Y_n^{-1} = A.$$

Notice that $\mathcal{CFB}_n(\Omega)$ is a similarity invariant set, by using the similarity orbit theorem, we can prove that

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Theorem

$\mathcal{CFB}_n(\Omega)$ is norm dense in $B_n(\Omega)$.

New progresses

Definition

Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$*. Define a Rosenblum operator* $\sigma_{T_1,T_2} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ *as*

$$\sigma_{T_1,T_2}(X) = T_1X - XT_2, \forall X \in \mathcal{L}(\mathcal{H}),$$

and a Rosenblum operator $\sigma_{T_1} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ *as*

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Definition (Property H)

Let $T \in \mathcal{CFB}_n(\Omega)$*. We call* T *satisfies the Property (H) if and only if the following statements hold: If* $Y \in B(\mathcal{H}_j, \mathcal{H}_i)$ *satisfies*

Then $Y = 0$*. That is equivalent to* $ker\sigma_{T_{i,i},T_{i+1,i+1}} \cap ran\sigma_{T_{i,i},T_{i+1,i+1}} = \{0\}$*.*

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(ii) $Y = T_{i,i}Z - ZT_{i+1,i+1}$, *for some* $Z, i < j = 1, \cdots, n$.

Then $Y = 0$*. That is equivalent to* $\ker\sigma_{T_{i,i},T_{i+1,i+1}} \cap \operatorname{ran}\sigma_{T_{i,i},T_{i+1,i+1}} = \{0\}$.

New progresses

Proposition

Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ *and* S_2 *be the right inverse of* T_2*. If* $\lim\limits_{n\to\infty} \frac{\|T_1^n\|\cdot\|S_2^n\|}{n} = 0$*, then the Property (H) holds i.e. If there exists* $X \in \mathcal{L}(\mathcal{H})$ *such that* $T_1X = XT_2$ *and* $X = T_1Y - YT_2$ *for some* Y*, then X=0.*

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Proposition

Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and S_2 be the right inverse of T_2. If $\lim\limits_{n\to\infty} \frac{\|T_1^n\|\cdot\|S_2^n\|}{n} = 0$, then the Property (H) holds i.e. If there exists $X \in \mathcal{L}(\mathcal{H})$ such that $T_1X = XT_2$ and $X = T_1Y - YT_2$ for some Y, then X=0.

Example

Let $A, B \in B_1(\mathbb{D})$ be backward shift operators with weighted sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$. If $\lim\limits_{n\to\infty} n^{\frac{\prod_{k=1}^{n} b_k}{\prod_{k=1}^{n} a_k}} = \infty$, then the Property (H) holds.

New progresses

Definition

We call $T \sim_{U+K} S$, *if there exists a unitary operator* U *and a compact operator* K *such that* $U + K$ *is invertible and* $(U + K)T = S(U + K)$.

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Lemma

Let $T, S \in B_1(\mathbb{D})$*, where* $S \sim_u (M_z^*, \mathcal{H}_{\mathcal{K}_S}, \mathcal{K}_S)$*, then we have*

$$T \sim_{U+K} S \Leftrightarrow K_S - K_T = \Delta ln\phi,$$

where ϕ *is a bounded function with*

$$\phi(w) = 1 + \frac{\sum\limits_{i=1}^{m} 2Ref_i(w)\bar{g}_i(w) + \sum\limits_{i=1}^{m} |g_i(w)|^2}{K_S(w,w)},$$

where m *is the rank of* K *and* $\{f_i\}_{i=1}^m, \{g_i\}_{i=1}^m \in \mathcal{H}_{\mathcal{K}_S}$ *are orthogonal sets,* $||f_i|| = 1, ||g_i|| \to 0$. *When* $K_S \geq K_T$*, then* $ln\phi$ *is subharmonic.*

New progresses

Proposition

Let $A, B \in B_1(\mathbb{D})$ *be backward weighted shift operators with weighted sequences* $\{a_k\}_{k=1}^{\infty}$ *and* $\{b_k\}_{k=1}^{\infty}$ *respectively. Then the following statements are equivalent:*

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(i) $A \sim_s B$ *implies* $A \sim_{U+K} B$,

New progresses

Proposition

Let $A, B \in B_1(\mathbb{D})$ be backward weighted shift operators with weighted sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ respectively. Then the following statements are equivalent:

(i) $A \sim_s B$ *implies* $A \sim_{U+K} B$,

(ii) $\lim\limits_{n\to\infty} \dfrac{\prod\limits_{k=1}^{n} a_k}{\prod\limits_{k=1}^{n} b_k}$ *exists and is not equal to zero.*

Similarity involving $U + K$

Main Theorem 1 [Jiang and Ji]

Let $T, \widetilde{T} \in \mathcal{C}FB_n(\Omega)$. Suppose the following statements hold

then we have

$$T \sim_s \widetilde{T} \Leftrightarrow \begin{cases} K_{T_i} - K_{\widetilde{T}_i} = \Delta ln\phi_i \\ \frac{\phi_i}{\phi_{i+1}}\theta_{i,i+1}(T) = \theta_{i,i+1}(\widetilde{T}) \end{cases}$$

where ϕ_i are the bounded subharmonic functions in the Lemma above.

Similarity involving $U + K$

Main Theorem 1 [Jiang and Ji]

Let $T, \widetilde{T} \in \mathcal{C}FB_n(\Omega)$. Suppose the following statements hold

(1) T and $\tilde{T}$ satisfy the Property (H);

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where ϕ_i are the bounded subharmonic functions in the Lemma above.

Similarity involving $U + K$

Main Theorem 1 [Jiang and Ji]

Let $T, \widetilde{T} \in \mathcal{C}FB_n(\Omega)$. Suppose the following statements hold

(1) T and $\tilde{T}$ satisfy the Property (H);

(2) $T_{i,i} \sim_s \tilde{T}_{i,i}$ implies $T_{i,i} \sim_{U+K} \tilde{T}_{i,i}$

then we have

$$T \sim_s \widetilde{T} \Leftrightarrow \begin{cases} K_{T_i} - K_{\widetilde{T}_i} = \Delta ln\phi_i \\ \frac{\phi_i}{\phi_{i+1}}\theta_{i,i+1}(T) = \theta_{i,i+1}(\widetilde{T}) \end{cases}$$

where ϕ_i are the bounded subharmonic functions in the Lemma above.

Strongly Property (H)

Definition (Strongly Property (H))

Let $T \in \mathcal{CFB}_n(\Omega)$. *We call* T *satisfies the strongly property (H) if and only if the following statements hold: If* $Y \in B(\mathcal{H}_j, \mathcal{H}_i)$ *satisfies*

Then $Y = 0$. *That is equivalent to* $ker\sigma_{T_{i,i},T_{j,j}} \cap ran\sigma_{T_{i,i},T_{j,j}} = \{0\}$.

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(i) $T_{i,i}Y = YT_{j,j}$,

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Let $T \in \mathcal{CFB}_n(\Omega)$. We call T satisfies the strongly property (H) if and only if the following statements hold: If $Y \in B(\mathcal{H}_j, \mathcal{H}_i)$ satisfies

(i) $T_{i,i}Y = YT_{j,j}$,

(ii) $Y = T_{i,i}Z - ZT_{j,j}$, *for some* $Z, i < j = 1, \cdots, n$.

Then $Y = 0$. That is equivalent to $ker\sigma_{T_{i,i},T_{j,j}} \cap ran\sigma_{T_{i,i},T_{j,j}} = \{0\}$.

Main Theorem 2

Main Theorem 2 [Jiang and Ji]

Let $T = ((T_{i,j}))_{n\times n}$ and $\tilde{T} = ((\tilde{T}_{i,j}))_{n\times n}$ be any two operators in $\mathcal{CFB}_n(\Omega)$, where $T_{i,j} = \tilde{T}_{i,j} = 0, i > j$. Suppose that T satisfies the strongly property (H). Then we have

$$T \sim_s \widetilde{T} \Leftrightarrow \begin{cases} X_i T_{i,i} = \tilde{T}_{i,i} X_i, \\ X_i T_{i,j} = \tilde{T}_{i,j} X_j, i = 1, 2, \cdots, n \end{cases}$$

where $X_i \in \mathcal{L}(\mathcal{H}_i, \tilde{\mathcal{H}}_i), i = 1, 2, \cdots, n$ are invertible operators.

Application

In the following theorem, Soumitra Ghara give a way to decide when an operator $T \in \mathcal{FB}_2(\mathbb{D})$ to be a weakly homogeneous operator.

Theorem (S. Ghara, Thesis,IISC, 2018)

Let $1 \leq \lambda \leq \mu \leq \lambda + 2$ and ψ be a non-zero function in $C(\bar{\mathbb{D}}) \cap Hol(\mathbb{D})$. The operator $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathcal{H}^{(\lambda)} \oplus \mathcal{H}^{(\mu)}$ is weakly homogeneous if and only if ψ is non-vanishing on $\bar{\mathbb{D}}$.

Although the description of the weakly homogeneous operators in $\mathcal{FB}_2(\mathbb{D})$ is more or less clear. However, the computation will become very difficult with the growth of the rank n.

Application

Thus, in general case, we need the intertwining operator between T and $\phi_\alpha(T), \alpha \in \mathbb{D}$ could be diagonal. That means we need to consider the operators in $\mathcal{CFB}_n(\mathbb{D})$ which satisfy the strongly Property (H). In the end of this talk, we will show that there also exists a lot examples of non-weakly homogeneous operators in $\mathcal{CFB}_n(\mathbb{D})$

Theorem 3[Jiang and Ji]

Let $T = \begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ 0 & T_{2,2} & T_{2,3} \\ 0 & 0 & T_{3,3} \end{pmatrix} \in \mathcal{CFB}_3(\mathbb{D})$. If T satisfies the strongly Property (H), then T is not weakly homogeneous.

End

Thank you!!

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