Quantifying Coherence via Quantum Uncertainty

Shunlong Luo (骆顺龙)

Academy of Math. & Systems Science
Chinese Academy of Sciences
luosl@amt.ac.cn

Mathematical and Physical Aspects of Information Sciences
Jan. 6-7, 2018, Harbin

Outline

1. Introduction

- The principle of superposition is a fundamental feature of quantum mechanics.

- Coherence arises form superposition, and is pivotal for quantum information processing.

- Coherece and decoherence are relative to measurement.

Quantum measurement:

- observable
- von Neumann measurement
- Lüders measurement
- quantum operation (channel, or positive operator valued measure)

Intimately related to measurements are:

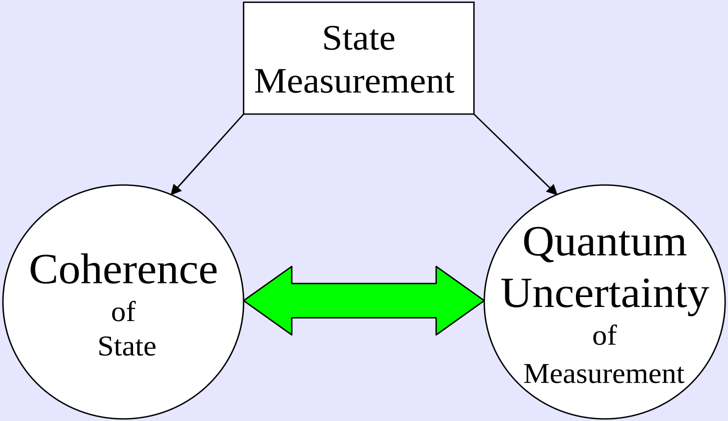
- uncertainties of measuring results
- coherence of quantum states

Problem:
What are the interrelations between coherence and uncertainty?

We will make, and elaborate on, the following identification:

Coherence = Quantum Uncertainty

S. Luo and Y. Sun, Quantum coherence versus quantum uncertainty, Phys. Rev. A **96**, 022130 (2017)



Coherence of ρ (w.r.t. M)
is identified with
quantum uncertainty of M (w.r.t. ρ)

2. Coherence

In recent years, flurry of interests in the quantification issues of coherence:

- Åberg, Quantifying superposition, arXiv: 0612146v1 (2006)
- Levi and Mintert, A quantitative theory of coherent delocalization, New J. Phys. **16**, 033007 (2014)
- Baumgratz *et al.*, Quantifying coherence, Phys. Rev. Lett. **113**, 140401 (2014)
- ······

Several important quantifiers of coherence:

- relative entropy of coherence (coherence cost)
- coherence formation
- robustness of coherence
- l^p-norm coherence
- distance based coherence
- $\cdots\cdots$

Resource theoretic perspective of coherence, in analogy to that of entanglement, has also been established.

A. Winter and D. Yang, Operational resource theory of coherence, Phys. Rev. Lett. **116**, 120404 (2016)

All these approaches are based on the notions of incoherent operations, which have many species such as

- maximally incoherent operations
- incoherent operations
- physically incoherent operations
- strictly incoherent operations
- genuinely incoherent operations
- dephase covariant operations
- translation invariant operations
- energy preserving operations
- ......

These diversities complicate the issue, and most of them are actually not free of coherence when implemented via ancillaries.

Another severe, neither necessary nor desirable, restriction of existent coherence measures lies in the reference bases, which are always taken as orthonormal bases, or equivalently, von Neumann measurements.

In both theoretical and practical investigations, it is necessary to consider: coherence w.r.t. general positive operator valued measures.

We will take a direct approach to coherence quantification via quantum uncertainty, which in turn is quantified via average quantum Fisher information.

We will elucidate that this viewpoint captures the essence of, leads to interesting implications for, and sheds considerable lights on, coherence.

3. Quantum Uncertainty

Given an observable A and a quantum state ρ, the variance

$$V(\rho, A) = \mathrm{tr}\rho A^2 - (\mathrm{tr}\rho A)^2$$

is a fundamental quantity representing the total uncertainty of A in ρ, which may be formally decomposed into a classical part and a quantum part as:
$V(\rho, A) = C(\rho, A) + Q(\rho, A).$

These two kinds of uncertainties are postulated to satisfy the following intuitive requirements:
(1). $Q(\rho, A)$ is convex in ρ. In contrast, the classical uncertainty $C(\rho, A)$ is concave in ρ.
(2). When ρ is pure, $V(\rho, A) = Q(\rho, A)$ and $C(\rho, A) = 0$. There is no classical mixing and all uncertainties are quantum for any pure state.

(3). When ρ commutes with A, $Q(\rho, A) = 0$ and $C(\rho, A) = V(\rho, A)$ because in this situation, ρ and A can be diagonalized simultaneously and thus behave like classical variables. Consequently there is no quantum uncertainty of A in ρ.

There is no unique choice of $Q(\rho, A)$, and depending on the context and problems, one may make different choices.

Based on the quantum estimation theory, it is natural to take quantum Fisher information as a measure of quantum uncertainty.

S. Luo, Classical versus quantum uncertainty, Theor. Math. Phys. **151**, 693 (2007)

Now for any measurement mathematically represented by a positive operator valued measure (POVM) $M = \{M_i : i = 1, 2, \cdots, m\}$ with $M_i \geq 0, \sum_i M_i = \mathbf{1}$, its action on a quantum state results in a post-measurement state $M(\rho) = \sum_i \sqrt{M_i}\rho\sqrt{M_i}$ in the non-selective case, and a post-measurement ensemble $\{\rho_i = \frac{1}{p_i}\sqrt{M_i}\rho\sqrt{M}_i, p_i = \mathrm{tr}\rho M_i\}$ in the selective case.

We define the total uncertainty of the measurement M in ρ as $V(\rho, M) = \sum_i V(\rho, M_i)$.

In order to extract the quantum part, we define the quantum uncertainty of the measurement M in ρ as

$$Q(\rho, M) = \sum_i F(\rho, M_i),$$

which indeed meets the above requirements. Here $F(\rho, M_i)$ is a version of quantum Fisher information of ρ w.r.t. M_i.

There are infinitely many versions of quantum Fisher information, among which two prominent ones are defined via symmetric logarithmic derivative and commutator, respectively. The latter corresponds to the Wigner-Yanase skew information

$$I(\rho, A) := -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, A]^2.$$

Meaning of skew information:

• Information content of ρ skew to A

• Non-commutativity between A and ρ

• Quantum Fisher information of ρ w.r.t. a parameter conjugate to A

• Quantum uncertainty of A in ρ

• Coherence of ρ w.r.t. A

• Asymmetry of ρ w.r.t. A

All these indicate that skew information is a significant and versatile quantity.

4. Coherence as Quantum Uncertainty

Measure of quantum uncertainty of $M = \{M_i\}$ in ρ:

$$Q(\rho, M) = \sum_{i=1}^{m} I(\rho, M_i).$$

Here the measurement M plays an active role, while the state ρ plays a passive role (i.e., serves as a background reference).

Taking a dual point of view, we regard the state ρ as active, and the measurement M as passive, and interpret this quantity as coherence of ρ w.r.t. M.
We remark that in the present context, there is no natural notions for incoherent states or incoherent operations.
It turns out that $Q(\rho, M)$ is indeed a bona fide measure for coherence, as consolidated by the following properties.

- The coherence is nonnegative, and vanishes if and only if ρ commutes with every M_i, i.e., $Q(\rho, M) \geq 0$, and the minimal value 0 is achieved if and only if $[\rho, M_i] = 0$.
- The coherence $Q(\rho, M)$ is convex in ρ, that is,

$$Q\Big(\sum_j c_j \rho_j, M\Big) \leq \sum_j c_j Q(\rho_j, M)$$

where $c_j \geq 0, \sum_j c_j = 1$, and ρ_j are quantum states.

• The coherence $Q(\rho, M)$ is unitary covariant in the sense that $Q(U\rho U^\dagger, UMU^\dagger) = Q(\rho, M)$ for any unitary operator U. Here $UMU^\dagger = \{UM_iU^\dagger\}$.

• The coherence $Q(\rho, M)$ is decreasing under partial trace in the sense that

$$Q(\rho^{ab}, M^a \otimes \mathbf{1}^b) \geq Q(\rho^a, M^a).$$

Here ρ^{ab} is a bipartite state, $M^a = \{M_i^a\}$ is a measurement on party a.

- The coherence $Q(\rho, M)$ is decreasing, i.e., $Q(\rho, M) \geq Q(\Phi(\rho), M)$ under any quantum operation Φ which does not disturb the measurement M (in the technical sense that $\Phi^\dagger(\sqrt{M_i}) = \sqrt{M_i}$, $\Phi^\dagger(M_i) = M_i$ for all i).

We emphasize that the present coherence measure is fundamentally different from existent measures of coherence:

• While previous coherence measures are w.r.t. to a fixed orthonormal base (equivalently, von Neumann measurement), here the coherence measure is more general, since it is constructed w.r.t. any quantum measurement. It is necessary to go beyond von Nuemann measurements in many situations.

• We do not rely on the notions of the so-called incoherent operations, which are very subtle and complicated.

• Unlike many other measures of coherence, here optimization is not involved.

We now sketch the proof of the last property. First, under the non-disturbance conditions, it follows that $I(\Phi(\rho), M_k) \leq I(\rho, M_k)$. To establish this, define the affinity:

$$A(\rho, \tau) := \mathrm{tr}\sqrt{\rho}\sqrt{\tau}.$$

Consider the von Neumann-Landau equation

$$i\frac{\partial}{\partial t}\rho_t = [M_k, \rho_t], \quad \rho_0 = \rho,$$

then $A(\rho_t, \rho) = 1 - I(\rho, M_k)t^2 + o(t^2)$ for sufficiently small t. Similarly, due to the non-disturbance condition, $\Phi(\rho_t)$ satisfies the same equation with generator M_k and initial condition $\Phi(\rho_t)|_{t=0} = \Phi(\rho)$, and thus

$$A(\Phi(\rho_t), \Phi(\rho)) = 1 - I(\Phi(\rho), M_k)t^2 + o(t^2)$$

for sufficiently small t.

Now by the monotonicity of affinity, we have

$$A(\Phi(\rho_t), \Phi(\rho)) \geq A(\rho_t, \rho_0)$$

which implies

$$I(\Phi(\rho), M_k) \leq I(\rho, M_k).$$

Summing these inequalities w.r.t. k yields the desired result.

Example

In order to illustrate quantum coherence w.r.t a general measurement, consider $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which is maximally coherent w.r.t. the standard base in an intuitive sense. Now consider the measurement $M = \{M_i : i = 1, 2\}$ with

$$M_1 = \gamma|0\rangle\langle 0| + (1-\gamma)|1\rangle\langle 1|,$$

$$M_2 = (1-\gamma)|0\rangle\langle 0| + \gamma|1\rangle\langle 1|$$

where $\gamma \in [0, 1]$ is a parameter.

Then direct evaluation shows that

$$Q(\rho, M) = 2\Big(\gamma - \frac{1}{2}\Big)^2,$$

which

- vanishes when $\gamma = \frac{1}{2}$, and
- achieves the maximum value $\frac{1}{2}$ when $\gamma = 0$ or 1.

Hence by adjusting γ, quantum coherence can take any value between 0 and $\frac{1}{2}$.

In many situations of probing (measuring) a quantum state, it is crucial to take a judicious trade-off between extracting information (which causes decoherence) and maintaining coherence, and general measurements beyond von Neumann measurements are necessary.

We recall that the most general state changes can be described by quantum operations with Kraus representations:

$$\Phi : \rho \to \{(p_i, \rho_i) : i \in I\},$$

which send an initial state to a quantum ensemble with $\rho_i = \frac{1}{p_i} \sum_k A_{ik} \rho A_{ik}^\dagger$, $p_i = \mathrm{tr} \sum_k A_{ik} \rho A_{ik}^\dagger$, $\sum_{ik} A_{ik}^\dagger A_{ik} = \mathbf{1}$, i.e., the measurement yields the outcome labeled by i with probability p_i, with the resulting post-measurement state ρ_i.

In this situation, how to define quantum uncertainty of this state change? Equivalently, how to define coherence of a state w.r.t. the most general measurement Φ?

This is an important and subtle issue. First, there is a corresponding POVM $M = \{M_i : i \in I\}$ with $M_i = \sum_k A_{ik}^\dagger A_{ik}$, and if we employ this measurement M to define the quantum uncertainty of Φ, we are reduced to the POVM case. However, it seems that such an approach misses many some intrinsic characteristics of Φ since the Kraus operators A_{ik} may not be Hermitian. It is desirable to extend the previous formalism and results to this general case, which is left as an open issue for further investigations.

• Specifying to Lüders Measurements

Consider a system with Hilbert space H. Let $\Pi = \{\Pi_i : i = 1, 2, \cdots, m\}$ be a Lüders measurement. This is equivalent to a direct sum decomposition:

$$H = \bigoplus_i H_i$$

with Π_i corresponding to the orthogonal projection onto the subspace $H_i = \Pi_i H$.

In particular, when all Π_i are one-dimensional, we have a von Neumann measurement, which is equivalent to an orthonormal base for H.

For the Lüders measurement $\Pi = \{\Pi_i\}$, the coherence has the following further nice properties.

- $Q(\rho, \Pi)$ can be alternatively expressed as

$$Q(\rho, \Pi) = \sum_{i \neq j} \mathrm{tr} \sqrt{\rho} \Pi_i \sqrt{\rho} \Pi_j$$

which is reminiscent of off-diagonal elements and interference, the characteristic features of coherence.

- $Q(\rho, \Pi)$ has the direct sum property in the sense that

$$Q\Big(\bigoplus_i \lambda_i \sigma_i, \Pi\Big) = \sum_i \lambda_i Q(\sigma_i, \Pi).$$

where $\lambda_i \geq 0, \sum_i \lambda_i = 1$, and σ_i are quantum states on H_i (thus *a priori* are quantum states on H).

• $Q(\rho,\Pi)$ has the following tensor product property:

$$\begin{aligned} & 1 - Q(\rho^a \otimes \rho^b, \Pi^{ab}) \\ = \ & (1 - Q(\rho^a, \Pi^a))(1 - Q(\rho^b, \Pi^b)), \end{aligned}$$

where ρ^{ab} is a bipartite state shared by parties a and b, $\Pi^a = \{\Pi_i^a\}$ and $\Pi^b = \{\Pi_j^b\}$ are Lüders measurements on parties a and b, respectively, and $\Pi^{ab} = \{\Pi_i^a \otimes \Pi_j^b\}$.

5. Maximal, Minimal, and Average Coherence

In general, $Q(\rho, \Pi)$ should be regarded as a functional of both ρ and Π.

Maximal coherence:

$$Q_{\max}(\rho) = \max_{\Pi} Q(\rho, \Pi)$$

Clearly, $Q_{\max}(\rho) = \max_U Q(\rho, U\Pi U^\dagger)$ where U is unitary.

It is natural to expect that $Q_{\max}(\rho)$ should be a measure of quantum information content, or quantum purity, of ρ. Indeed, we have

$$Q_{\max}(\rho) = \frac{1}{n}\sum_{j=1}^{n^2} I(\rho, X_j) = 1 - \frac{1}{n}(\mathrm{tr}\sqrt{\rho})^2$$

where $\{X_i\}$ is an ONB for $L(H)$ (operators on H with $\langle A|B\rangle = \mathrm{tr}A^\dagger B$), and $n = \dim H$. This indicates that quantum purity may be interpreted as maximal coherence.

When the worst cases are relevant, one may be interested in the minimal coherence

$$Q_{\min}(\rho) = \min_{\Pi} Q(\rho, \Pi).$$

In particular, quantum discord may be interpreted as the minimal coherence, with minimization over all local von Neumann measurements.

To illustrate this, consider a bipartite state ρ^{ab} shared by two parties *a* and *b*, and let $\Pi^a = \{\Pi_i^a\}$ be a von Neumann measurement on party *a*, then $\Pi^a \otimes \mathbf{1}^b = \{\Pi_i^a \otimes \mathbf{1}^b\}$ is a Lüders measurement on the combined system *ab*.

The geometric discord

$$D_{\mathrm{H}}(\rho^{ab}) := \min_{\Pi^a} \mathrm{tr}(\sqrt{\rho^{ab}} - (\Pi^a \otimes \mathbf{1}^b)(\sqrt{\rho^{ab}}))^2$$

quantifies quantum correlations (w.r.t. party a) in ρ^{ab}, where

$$(\Pi^a \otimes \mathbf{1}^b)(\sqrt{\rho^{ab}}) = \sum_i (\Pi_i^a \otimes \mathbf{1}^b)\sqrt{\rho^{ab}}(\Pi_i^a \otimes \mathbf{1}^b).$$

On the other hand,
$\min_{\Pi^a} Q(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = D_{\mathrm{H}}(\rho^{ab})$.
Thus the geometric discord is precisely the minimal coherence.

Intermediate between maximal and minimal coherence is the average coherence

$$Q_{\rm ave}(\rho) = \int_{\mathcal{U}} Q(U\rho U^\dagger, \Pi) dU.$$

Here the integration is w.r.t. the Haar measure on the group of unitary operators. The explicit evaluation of the integral remains to be investigated.

The coherence measure $Q(\rho, \Pi)$ should be compared with the K-coherence $I(\rho, K)$, which violates the important axiom for monotonicity. $Q(\rho, \Pi)$ satisfies the monotonicity.

D. Girolami, Observable measure of quantum coherence in finite dimensional systems, Phys. Rev. Lett. **113**, 170401 (2014).
S. Luo, Y. Sun, Partial coherence with application to the monotonicity problem of coherence involving skew information, Phys. Rev. A **96**, 022136 (2017)

6. Summary

In contrast to the resource theoretic approach to coherence, we have

- introduced a direct and intuitive approach to coherence by identifying quantum uncertainty and coherence,
- have defined the corresponding coherence measure via quantum Fisher information.

The coherence measure is defined, without reference to incoherent operations, in a more broad framework involving general positive operator valued measures rather than orthonormal bases corresponding to von Neumann measurements, and it enjoys several desirable and intuitive properties.

In particular, we are led to corroborate the following formal identifications:

- Quantum Uncertainty $\sim$ Coherence
- Quantum purity $\sim$ Maximal Coherence
- Quantum Discord $\sim$ Minimal Coherence

A lot of important questions call for further investigations including

- foundational implications,
- operational significance,
- resource theoretic connection, and
- experimental usage

of the ideas and results illustrated here.

Thank you!