The Khintchine inequality

Piotr Nayar

Institute of Mathematics
University of Warsaw

19/10/2022

Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. symmetric Bernoulli random variables,

$$\mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2}, \qquad i = 1, \ldots, n.$$

Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. symmetric Bernoulli random variables,

$$\mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2}, \qquad i = 1, \ldots, n.$$

We consider the sum

$$S = \sum_{i=1}^{n} a_i \varepsilon_i, \qquad a_i \in \mathbb{R}.$$

Let $\varepsilon_1, \ldots, \varepsilon_n$ be i.i.d. symmetric Bernoulli random variables,

$$\mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2}, \qquad i = 1, \ldots, n.$$

We consider the sum

$$S = \sum_{i=1}^{n} a_i \varepsilon_i, \qquad a_i \in \mathbb{R}.$$

Theorem (A. Khintchin, 1923, J. Littlewood, 1930)

For $p > q > 0$ there exists a constant $C_{p,q}$ depending only on p, q such that

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q} (\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

Let $\varepsilon_1, \dots, \varepsilon_n$ be i.i.d. symmetric Bernoulli random variables,

$$\mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2}, \qquad i = 1, \dots, n.$$

We consider the sum

$$S = \sum_{i=1}^{n} a_i \varepsilon_i, \qquad a_i \in \mathbb{R}.$$

Theorem (A. Khintchin, 1923, J. Littlewood, 1930)

For $p > q > 0$ there exists a constant $C_{p,q}$ depending only on p, q such that

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q} (\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

Note that $\mathbb{E}S^2 = \sum_{i=1}^n a_i^2$.

Applications

Applications

- probability theory

Applications

- probability theory
- functional analysis (Banach space theory)

Applications

- probability theory
- functional analysis (Banach space theory)
- convex geometry

Applications

- probability theory
- functional analysis (Banach space theory)
- convex geometry
- combinatorics

Applications

- probability theory
- functional analysis (Banach space theory)
- convex geometry
- combinatorics
- theoretical computer science

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

$$A_{p,q} := \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}}, \quad n \to \infty$$

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \le C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

$$A_{p,q} := \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}}, \quad n \to \infty$$

$$B_{p,q} := \frac{\left(\mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right|^p\right)^{\frac{1}{p}}}{\left(\mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right|^q\right)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}.$$

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \le C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

$$A_{p,q} := \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}}, \quad n \to \infty$$

$$B_{p,q} := \frac{(\mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right|^p)^{\frac{1}{p}}}{(\mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right|^q)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}.$$

We always have $C_{p,q} \ge \max(A_{p,q}, B_{p,q})$

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

$$A_{p,q} := \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}}, \quad n \to \infty$$

$$B_{p,q} := \frac{\left(\mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right|^p\right)^{\frac{1}{p}}}{\left(\mathbb{E}\left|\frac{\varepsilon_1+\varepsilon_2}{\sqrt{2}}\right|^q\right)^{\frac{1}{q}}}, \qquad S = \frac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}.$$

We always have $C_{p,q} \geq \max(A_{p,q}, B_{p,q})$

In known cases: $C_{p,q} = \max(A_{p,q}, B_{p,q})$

Optimal constants

$$\left(\mathbb{E}|S|^p\right)^{\frac{1}{p}} \leq C_{p,q}\left(\mathbb{E}|S|^q\right)^{\frac{1}{q}}.$$

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

- $C_{p,2} = A_{p,2}$ for $p \geq 2$ (Whittle '60, Steckin '61, Haagerup '82)

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

- $C_{p,2} = A_{p,2}$ for $p \geq 2$ (Whittle '60, Steckin '61, Haagerup '82)
- There exists $q_0 \approx 1.847$ (given by equation $A_{2,q} = B_{2,q}$) such that

$$C_{2,q} = \begin{cases} B_{2,q} & 0 < q < p_0 \\ A_{2,q} & p_0 \leq q \leq 2 \end{cases}$$

(Szarek '78, Haagerup '82)

Optimal constants

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \le C_{p,q}(\mathbb{E}|S|^q)^{\frac{1}{q}}.$$

- $C_{p,2} = A_{p,2}$ for $p \ge 2$ (Whittle '60, Steckin '61, Haagerup '82)
- There exists $q_0 \approx 1.847$ (given by equation $A_{2,q} = B_{2,q}$) such that
$$C_{2,q} = \begin{cases} B_{2,q} & 0 < q < p_0 \\ A_{2,q} & p_0 \le q \le 2 \end{cases}$$
(Szarek '78, Haagerup '82)
- $C_{p,q} = A_{p,q}$ if p, q are even integers
(Czerwiński '08, N.-Oleszkiewicz '12)

$C_{p,2}$ for $p \geq 3$ (relatively easy)

Assume $\sum_{i=1}^n a_i^2 = 1$. The goal is to prove that

$$\mathbb{E}|S|^p \leq \mathbb{E}|G|^p, \qquad (\mathbb{E}|S|^p)^{1/p} \leq \frac{(\mathbb{E}|G|^p)^{1/p}}{(\mathbb{E}|G|^2)^{1/2}} (\mathbb{E}|S|^2)^{1/2}$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

Assume $\sum_{i=1}^n a_i^2 = 1$. The goal is to prove that

$$\mathbb{E}|S|^p \leq \mathbb{E}|G|^p, \qquad (\mathbb{E}|S|^p)^{1/p} \leq \frac{(\mathbb{E}|G|^p)^{1/p}}{(\mathbb{E}|G|^2)^{1/2}} (\mathbb{E}|S|^2)^{1/2}$$

Let S' be an independent copy of S. Then

$$\frac{S + S'}{\sqrt{2}} = \sum_{i=1}^n a_i \frac{\varepsilon_i + \varepsilon_i'}{\sqrt{2}} = \sum_{i=1}^n a_i X_i, \qquad \mathbb{E}X_i^2 = 1.$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

Assume $\sum_{i=1}^n a_i^2 = 1$. The goal is to prove that

$$\mathbb{E}|S|^p \leq \mathbb{E}|G|^p, \qquad (\mathbb{E}|S|^p)^{1/p} \leq \frac{(\mathbb{E}|G|^p)^{1/p}}{(\mathbb{E}|G|^2)^{1/2}}(\mathbb{E}|S|^2)^{1/2}$$

Let S' be an independent copy of S. Then

$$\frac{S+S'}{\sqrt{2}} = \sum_{i=1}^n a_i \frac{\varepsilon_i + \varepsilon_i'}{\sqrt{2}} = \sum_{i=1}^n a_i X_i, \qquad \mathbb{E}X_i^2 = 1.$$

We want to apply the following scheme:

$$\mathbb{E}|S|^p \leq \mathbb{E}\left|\frac{S+S'}{\sqrt{2}}\right|^p \leq \mathbb{E}\left|\frac{\frac{S+S'}{\sqrt{2}} + \frac{S''+S'''}{\sqrt{2}}}{\sqrt{2}}\right|^p = \mathbb{E}\left|\frac{S+S'+S''+S'''}{\sqrt{4}}\right|$$

$$\leq \ldots \left|\frac{\sum_{i=1}^{2^n} S_i}{\sqrt{2^n}}\right| \xrightarrow[n\to\infty]{} \mathbb{E}|G|^p.$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

The inequality

$$\mathbb{E}\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|^p = \mathbb{E}|S|^p \leq \mathbb{E}\left|\frac{S+S'}{\sqrt{2}}\right|^p = \mathbb{E}\left|\sum_{i=1}^{n} a_i X_i\right|^p$$

is a consequence of the following fact

$C_{p,2}$ for $p \geq 3$ (relatively easy)

The inequality

$$\mathbb{E}\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|^p = \mathbb{E}|S|^p \leq \mathbb{E}\left|\frac{S+S'}{\sqrt{2}}\right|^p = \mathbb{E}\left|\sum_{i=1}^{n} a_i X_i\right|^p$$

is a consequence of the following fact

Fact

If $p \geq 3$ and if $X_1, \ldots, X_n$ are symmetric random variables satisfying $\mathbb{E}X_i^2 = 1$, then

$$\mathbb{E}\left|\sum_{i=1}^{n} a_i X_i\right|^p \geq \mathbb{E}\left|\sum_{i=1}^{n} a_i \varepsilon_i\right|^p$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

In fact we want to exchange $X_i \to \varepsilon_i$ one by one:

$$\mathbb{E}|aX + b|^p \geq \mathbb{E}|a\varepsilon + b|^p, \qquad \mathbb{E}X^2 = 1.$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

In fact we want to exchange $X_i \to \varepsilon_i$ one by one:

$$\mathbb{E}|aX + b|^p \geq \mathbb{E}|a\varepsilon + b|^p, \qquad \mathbb{E}X^2 = 1.$$

The function

$$f(t) = \mathbb{E}_\varepsilon |a\varepsilon\sqrt{t} + b|^p$$

is convex. In fact we can assume $a = b = 1$ and then

$$f(t) = \frac{1}{2}(|1 + \sqrt{t}|^p + |1 - \sqrt{t}|^p).$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

In fact we want to exchange $X_i \to \varepsilon_i$ one by one:

$$\mathbb{E}|aX + b|^p \geq \mathbb{E}|a\varepsilon + b|^p, \qquad \mathbb{E}X^2 = 1.$$

The function

$$f(t) = \mathbb{E}_\varepsilon |a\varepsilon\sqrt{t} + b|^p$$

is convex. In fact we can assume $a = b = 1$ and then

$$f(t) = \frac{1}{2}(|1 + \sqrt{t}|^p + |1 - \sqrt{t}|^p).$$

Thus, since $X \sim \varepsilon|X|$

$$\mathbb{E}|aX + b|^p = \mathbb{E}|a\varepsilon|X| + b|^p = \mathbb{E}|a\varepsilon\sqrt{X^2} + b|^p = \mathbb{E}f(X^2)$$
$$\geq f(\mathbb{E}X^2) = f(1) = \mathbb{E}|a\varepsilon + b|^p.$$

$C_{p,2}$ for $p \geq 3$ (relatively easy)

In fact we want to exchange $X_i \to \varepsilon_i$ one by one:

$$\mathbb{E}|aX + b|^p \geq \mathbb{E}|a\varepsilon + b|^p, \qquad \mathbb{E}X^2 = 1.$$

The function

$$f(t) = \mathbb{E}_\varepsilon |a\varepsilon\sqrt{t} + b|^p$$

is convex. In fact we can assume $a = b = 1$ and then

$$f(t) = \frac{1}{2}(|1 + \sqrt{t}|^p + |1 - \sqrt{t}|^p).$$

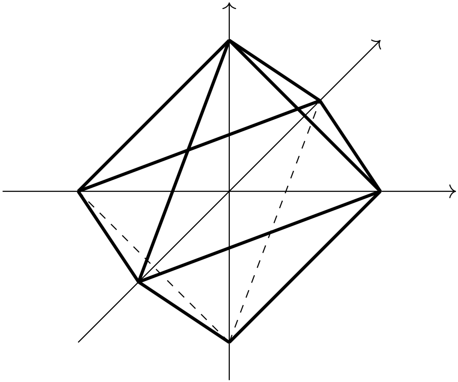
Thus, since $X \sim \varepsilon|X|$

$$\mathbb{E}|aX + b|^p = \mathbb{E}|a\varepsilon|X| + b|^p = \mathbb{E}|a\varepsilon\sqrt{X^2} + b|^p = \mathbb{E}f(X^2)$$
$$\geq f(\mathbb{E}X^2) = f(1) = \mathbb{E}|a\varepsilon + b|^p.$$

Question: is it true that $\mathbb{E}\left|\frac{S+S'}{\sqrt{2}}\right|^p \geq \mathbb{E}|S|^p$ for $p \in (2,3)$?

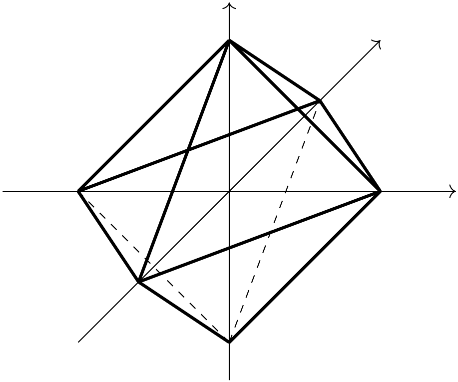
$C_{2,1}$ – geometric interpretation

$$\text{Cross-polytope} = B_1^n = \{x \in \mathbb{R}^n : |x_1| + \ldots + |x_n| \leq 1\}$$



$C_{2,1}$ – geometric interpretation

$$\text{Cross-polytope} \;=\; B_1^n \;=\; \{x \in \mathbb{R}^n : \; |x_1| + \ldots + |x_n| \leq 1\}$$



$$|\mathrm{Proj}_{a^\perp} B_1^n| = \frac{1}{2} \cdot |F| \cdot \sum_{\varepsilon \in \{-1,1\}^n} |\langle a, \varepsilon \rangle| = 2^{n-1} |F| \cdot \mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|$$

Walsh system

Walsh system

- Take $Q_n = \{-1, 1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.

Walsh system

- Take $Q_n = \{-1, 1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$

Walsh system

- Take $Q_n = \{-1, 1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$
- Scalar product $\langle f, g \rangle = \mathbb{E}fg$.

Walsh system

- Take $Q_n = \{-1,1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$
- Scalar product $\langle f, g \rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i \in S} x_i$.

Walsh system

- Take $Q_n = \{-1,1\}^n$ with uniform measure and define $L_2(Q_n,\mu)$.
- $\mathbb{E}f = 2^{-n}\sum_{x\in Q_n} f(x)$
- Scalar product $\langle f,g\rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i\in S} x_i$.
- We have $\mathbb{E}w_S = \delta_{S=\emptyset}$

Walsh system

- Take $Q_n = \{-1,1\}^n$ with uniform measure and define $L_2(Q_n,\mu)$.
- $\mathbb{E}f = 2^{-n}\sum_{x\in Q_n} f(x)$
- Scalar product $\langle f,g\rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i\in S} x_i$.
- We have $\mathbb{E}w_S = \delta_{S=\emptyset}$
- Note that $w_S w_T = w_{S\Delta T}$ and thus $\langle w_S, w_T\rangle = \delta_{S=T}$

Walsh system

- Take $Q_n = \{-1, 1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$
- Scalar product $\langle f, g \rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i \in S} x_i$.
- We have $\mathbb{E}w_S = \delta_{S=\emptyset}$
- Note that $w_S w_T = w_{S \Delta T}$ and thus $\langle w_S, w_T \rangle = \delta_{S=T}$
- Therefore w_S form an orthonormal basis.

Walsh system

- Take $Q_n = \{-1, 1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$
- Scalar product $\langle f, g \rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i \in S} x_i$.
- We have $\mathbb{E}w_S = \delta_{S=\emptyset}$
- Note that $w_S w_T = w_{S \Delta T}$ and thus $\langle w_S, w_T \rangle = \delta_{S=T}$
- Therefore w_S form an orthonormal basis.
- Any function can be written as $f = \sum_S a_S w_S$

Walsh system

- Take $Q_n = \{-1, 1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$
- Scalar product $\langle f, g \rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i \in S} x_i$.
- We have $\mathbb{E}w_S = \delta_{S=\emptyset}$
- Note that $w_S w_T = w_{S \Delta T}$ and thus $\langle w_S, w_T \rangle = \delta_{S=T}$
- Therefore w_S form an orthonormal basis.
- Any function can be written as $f = \sum_S a_S w_S$
- If $f = \sum_S a_S w_S$, $g = \sum_S b_S w_S$ then $\langle f, g \rangle = \sum_S a_S b_S$.

Walsh system

- Take $Q_n = \{-1,1\}^n$ with uniform measure and define $L_2(Q_n, \mu)$.
- $\mathbb{E}f = 2^{-n} \sum_{x \in Q_n} f(x)$
- Scalar product $\langle f, g \rangle = \mathbb{E}fg$.
- Walsh functions: for $S \subseteq [n]$ take $w_S(x) = \prod_{i \in S} x_i$.
- We have $\mathbb{E}w_S = \delta_{S=\emptyset}$
- Note that $w_S w_T = w_{S \Delta T}$ and thus $\langle w_S, w_T \rangle = \delta_{S=T}$
- Therefore w_S form an orthonormal basis.
- Any function can be written as $f = \sum_S a_S w_S$
- If $f = \sum_S a_S w_S$, $g = \sum_S b_S w_S$ then $\langle f, g \rangle = \sum_S a_S b_S$.
- $\mathrm{Var}(f) = \mathbb{E}f^2 - (\mathbb{E}f)^2 = \sum_S a_S^2 - a_\emptyset^2 = \sum_{S \neq \emptyset} a_S^2$

Poincaré inequality

$$(Lf)(x) = \frac{1}{2}\sum_{y \sim x}(f(y) - f(x)), \qquad Lw_S = -|S|w_S$$

Poincaré inequality

$$(Lf)(x) = \frac{1}{2}\sum_{y\sim x}(f(y) - f(x)), \qquad Lw_S = -|S|w_S$$

Fact

For any $f : Q_n \to \mathbb{R}$ we have

$$\mathrm{Var}(f) \leq \mathbb{E}(-Lf)f, \qquad \mathrm{Var}(f) \leq \frac{1}{2}\mathbb{E}(-Lf)f \quad [f - even]$$

Poincaré inequality

$$(Lf)(x) = \frac{1}{2}\sum_{y\sim x}(f(y) - f(x)), \qquad Lw_S = -|S|w_S$$

Fact

For any $f : Q_n \to \mathbb{R}$ we have

$$\mathrm{Var}(f) \leq \mathbb{E}(-Lf)f, \qquad \mathrm{Var}(f) \leq \frac{1}{2}\mathbb{E}(-Lf)f \quad [f - even]$$

Proof. We have $\mathrm{Var}(f) = \sum_{|S|\geq 1} a_S^2$ and

$$\mathbb{E}(-Lf)f = \left\langle \sum a_S|S|w_S, \sum a_S w_S \right\rangle = \sum_S |S|a_S^2.$$

f even $\implies a_S = 0$ when $|S| = 1$, since $a_{\{i\}} = \mathbb{E}f(x)x_i = 0$,

$$\mathbb{E}(-Lf)f = \sum_{|S|\geq 2} |S|a_S^2$$

$C_{2,1} = \sqrt{2}$ (proof of Latała and Oleszkiewicz)

Define $f : Q_n \to \mathbb{R}$ via

$$f(x) = \left\| \sum_{i=1}^{n} v_i x_i \right\|, \qquad v_i \in V.$$

We want to prove the inequality $(\mathbb{E}f^2)^{\frac{1}{2}} \leq \sqrt{2}\mathbb{E}f$ or equivalently $\mathbb{E}f^2 \leq 2(\mathbb{E}f)^2$.

$C_{2,1} = \sqrt{2}$ (proof of Latała and Oleszkiewicz)

Define $f : Q_n \to \mathbb{R}$ via

$$f(x) = \left\| \sum_{i=1}^{n} v_i x_i \right\|, \qquad v_i \in V.$$

We want to prove the inequality $(\mathbb{E}f^2)^{\frac{1}{2}} \leq \sqrt{2}\mathbb{E}f$ or equivalently $\mathbb{E}f^2 \leq 2(\mathbb{E}f)^2$. We claim that it is enough to show that $(-Lf) \leq f$. Indeed, we then have

$$\mathbb{E}f^2 - (\mathbb{E}f)^2 = \mathrm{Var}(f) \leq \frac{1}{2}\mathbb{E}(-Lf)f \leq \frac{1}{2}\mathbb{E}f^2$$

$C_{2,1} = \sqrt{2}$ (proof of Latała and Oleszkiewicz)

Define $f : Q_n \to \mathbb{R}$ via

$$f(x) = \left\| \sum_{i=1}^{n} v_i x_i \right\|, \qquad v_i \in V.$$

We want to prove the inequality $(\mathbb{E}f^2)^{\frac{1}{2}} \leq \sqrt{2}\mathbb{E}f$ or equivalently $\mathbb{E}f^2 \leq 2(\mathbb{E}f)^2$. We claim that it is enough to show that $(-Lf) \leq f$. Indeed, we then have

$$\mathbb{E}f^2 - (\mathbb{E}f)^2 = \mathrm{Var}(f) \leq \frac{1}{2}\mathbb{E}(-Lf)f \leq \frac{1}{2}\mathbb{E}f^2$$

It suffices to observe that

$$(-Lf)(x) = \frac{1}{2}\sum_{y \sim x}(f(x) - f(y)) = \frac{n}{2}f(x) - \frac{1}{2}\sum_{y \sim x}\left\| \sum_{i=1}^{n} v_i y_i \right\|$$

$$\leq \frac{n}{2}f(x) - \frac{1}{2}\left\| \sum_{i=1}^{n} v_i \sum_{y \sim x} y_i \right\| = \frac{n}{2}f(x) - \frac{n-2}{2}\left\| \sum_{i=1}^{n} v_i x_i \right\| = f(x).$$

Symmetric polynomials

For real numbers $c_1, c_2, \ldots,$ we define

$$\sigma_k^{(n)} = \sum_{S \subseteq [n], |S|=k} \prod_{i \in S} c_i, \qquad \sigma_k = \sum_{S \subseteq \mathbb{N}, |S|=k} \prod_{i \in S} c_i$$

Symmetric polynomials

For real numbers $c_1, c_2, \ldots,$ we define

$$\sigma_k^{(n)} = \sum_{S \subseteq [n], |S| = k} \prod_{i \in S} c_i, \qquad \sigma_k = \sum_{S \subseteq \mathbb{N}, |S| = k} \prod_{i \in S} c_i$$

Newton inequalities:

$$\left(\frac{\sigma_k^{(n)}}{\binom{n}{k}} \right)^2 \geq \frac{\sigma_{k+1}^{(n)}}{\binom{n}{k+1}} \cdot \frac{\sigma_{k-1}^{(n)}}{\binom{n}{k-1}}, \qquad (k!\sigma_k)^2 \geq ((k+1)!\sigma_{k+1}) \cdot ((k-1)!\sigma_{k-1})$$

Symmetric polynomials

For real numbers $c_1, c_2, \ldots,$ we define

$$\sigma_k^{(n)} = \sum_{S \subseteq [n], |S|=k} \prod_{i \in S} c_i, \qquad \sigma_k = \sum_{S \subseteq \mathbb{N}, |S|=k} \prod_{i \in S} c_i$$

Newton inequalities:

$$\left(\frac{\sigma_k^{(n)}}{\binom{n}{k}} \right)^2 \geq \frac{\sigma_{k+1}^{(n)}}{\binom{n}{k+1}} \cdot \frac{\sigma_{k-1}^{(n)}}{\binom{n}{k-1}}, \qquad (k!\sigma_k)^2 \geq ((k+1)!\sigma_{k+1}) \cdot ((k-1)!\sigma_{k-1})$$

A sequence (b_k) is called log-concave if $b_k^2 \geq b_{k+1} b_{k-1}$, $k \geq 1$.
The sequences

$$b_k^{(n)} = \frac{\sigma_k^{(n)}}{\binom{n}{k}}, \qquad b_k = k!\sigma_k$$

are log-concave.

Newton inequality - proof

Take the real rooted polynomial

$$P(x) = (1 + c_1 x) \dots (1 + c_n x) = \sum_{k=0}^{n} \sigma_k^{(n)} x^k$$

Operations $P(x) \to P^{(l)}(x)$ and $P(x) \to x^n P(x^{-1})$ preserve real-rootedness.

Newton inequality - proof

Take the real rooted polynomial

$$P(x) = (1 + c_1 x) \dots (1 + c_n x) = \sum_{k=0}^{n} \sigma_k^{(n)} x^k$$

Operations $P(x) \to P^{(l)}(x)$ and $P(x) \to x^n P(x^{-1})$ preserve real-rootedness.

$$\sum_{k=0}^{n} \sigma_k^{(n)} x^k \xrightarrow{\partial^{j-1}} \sum_{k \geq j-1} \sigma_k^{(n)} \frac{k!}{(k-j+1)!} x^{k-j+1} \xrightarrow{x^{n-j+1} f(x^{-1})}$$

$$\sum_{k \geq j-1} \frac{\sigma_k^{(n)} k!}{(k-j+1)!} x^{n-k} \xrightarrow{\partial^{n-j-1}} \sum_{k \in \{j-1, j, j+1\}} \frac{\sigma_k^{(n)} k! (n-k)! x^{j-k+1}}{(k-j+1)!(j-k+1)!}$$

$$= \frac{1}{2} \tau_{j-1} + \tau_j x + \frac{1}{2} \tau_{j+1} x^2, \qquad \tau_j = \sigma_j^{(n)} j! (n-j)! = \frac{\sigma_j^{(n)}}{\binom{n}{j}} \cdot n!$$

Newton inequality - proof

Take the real rooted polynomial

$$P(x) = (1 + c_1 x) \dots (1 + c_n x) = \sum_{k=0}^{n} \sigma_k^{(n)} x^k$$

Operations $P(x) \to P^{(l)}(x)$ and $P(x) \to x^n P(x^{-1})$ preserve real-rootedness.

$$\sum_{k=0}^{n} \sigma_k^{(n)} x^k \xrightarrow{\partial^{j-1}} \sum_{k \geq j-1} \sigma_k^{(n)} \frac{k!}{(k-j+1)!} x^{k-j+1} \xrightarrow{x^{n-j+1} f(x^{-1})}$$

$$\sum_{k \geq j-1} \frac{\sigma_k^{(n)} k!}{(k-j+1)!} x^{n-k} \xrightarrow{\partial^{n-j-1}} \sum_{k \in \{j-1, j, j+1\}} \frac{\sigma_k^{(n)} k!(n-k)! x^{j-k+1}}{(k-j+1)!(j-k+1)!}$$

$$= \frac{1}{2}\tau_{j-1} + \tau_j x + \frac{1}{2}\tau_{j+1} x^2, \qquad \tau_j = \sigma_j^{(n)} j!(n-j)! = \frac{\sigma_j^{(n)}}{\binom{n}{j}} \cdot n!$$

$$\Delta \geq 0 \implies \tau_j^2 \geq \tau_{j-1}\tau_{j+1}$$

$C_{p,q} = (\mathbb{E}|G|^p)^{1/p}/(\mathbb{E}|G|^q)^{1/q}$ (proof of N. and Tkocz)

For $S = \sum_{i=1}^n a_i \varepsilon_i$ we want to show

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}} (\mathbb{E}|S|^q)^{\frac{1}{q}}$$

$C_{p,q} = (\mathbb{E}|G|^p)^{1/p}/(\mathbb{E}|G|^q)^{1/q}$ (proof of N. and Tkocz)

For $S = \sum_{i=1}^n a_i\varepsilon_i$ we want to show

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}(\mathbb{E}|S|^q)^{\frac{1}{q}}$$

Equivalently

$$\frac{(\mathbb{E}|S|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^p)^{\frac{1}{p}}} \leq \frac{(\mathbb{E}|S|^q)^{\frac{1}{q}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}$$

$C_{p,q} = (\mathbb{E}|G|^p)^{1/p}/(\mathbb{E}|G|^q)^{1/q}$ (proof of N. and Tkocz)

For $S = \sum_{i=1}^n a_i\varepsilon_i$ we want to show

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \le \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}(\mathbb{E}|S|^q)^{\frac{1}{q}}$$

Equivalently

$$\frac{(\mathbb{E}|S|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^p)^{\frac{1}{p}}} \le \frac{(\mathbb{E}|S|^q)^{\frac{1}{q}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}$$

In other words

$$b_k^{1/k} \searrow \qquad b_k = \frac{\mathbb{E}|S|^{2k}}{\mathbb{E}|G|^{2k}}, \qquad \frac{\log b_k}{k} \searrow$$

$C_{p,q} = (\mathbb{E}|G|^p)^{1/p}/(\mathbb{E}|G|^q)^{1/q}$ (proof of N. and Tkocz)

For $S = \sum_{i=1}^n a_i\varepsilon_i$ we want to show

$$(\mathbb{E}|S|^p)^{\frac{1}{p}} \leq \frac{(\mathbb{E}|G|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}(\mathbb{E}|S|^q)^{\frac{1}{q}}$$

Equivalently

$$\frac{(\mathbb{E}|S|^p)^{\frac{1}{p}}}{(\mathbb{E}|G|^p)^{\frac{1}{p}}} \leq \frac{(\mathbb{E}|S|^q)^{\frac{1}{q}}}{(\mathbb{E}|G|^q)^{\frac{1}{q}}}$$

In other words

$$b_k^{1/k} \searrow \qquad b_k = \frac{\mathbb{E}|S|^{2k}}{\mathbb{E}|G|^{2k}}, \qquad \frac{\log b_k}{k} \searrow$$

It is enough to prove

$$\frac{\log b_{k+1} + \log b_{k-1}}{2} \leq \log b_k, \qquad b_k^2 \geq b_{k+1}b_{k-1}.$$

$$\mathbb{E}e^{\sqrt{2x}S} = \sum_{k\geq 0} \frac{\sqrt{2x}^k}{k!}\mathbb{E}S^k = \sum_{k\geq 0} \frac{\sqrt{2x}^{2k}}{(2k)!}\mathbb{E}S^{2k}$$

$$= \sum_{k\geq 0} \frac{2^k x^k}{(2k-1)!!2^k k!}\mathbb{E}S^{2k} = \sum_{k\geq 0} \frac{x^k}{k!} \cdot \frac{\mathbb{E}S^{2k}}{\mathbb{E}G^{2k}} = \sum_{k\geq 0} b_k \frac{x^k}{k!}$$

$$\mathbb{E}e^{\sqrt{2x}S} = \sum_{k\geq 0} \frac{\sqrt{2x}^k}{k!}\mathbb{E}S^k = \sum_{k\geq 0} \frac{\sqrt{2x}^{2k}}{(2k)!}\mathbb{E}S^{2k}$$

$$= \sum_{k\geq 0} \frac{2^k x^k}{(2k-1)!!2^k k!}\mathbb{E}S^{2k} = \sum_{k\geq 0} \frac{x^k}{k!} \cdot \frac{\mathbb{E}S^{2k}}{\mathbb{E}G^{2k}} = \sum_{k\geq 0} b_k \frac{x^k}{k!}$$

On the other hand

$$\mathbb{E}e^{\sqrt{2x}S} = \mathbb{E}\prod_{i=1}^n e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^n \mathbb{E}e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^n \cosh\left(\sqrt{2x}a_i\right)$$

$$\mathbb{E}e^{\sqrt{2x}S} = \sum_{k\geq 0} \frac{\sqrt{2x}^k}{k!}\mathbb{E}S^k = \sum_{k\geq 0} \frac{\sqrt{2x}^{2k}}{(2k)!}\mathbb{E}S^{2k}$$

$$= \sum_{k\geq 0} \frac{2^k x^k}{(2k-1)!!2^k k!}\mathbb{E}S^{2k} = \sum_{k\geq 0} \frac{x^k}{k!}\cdot\frac{\mathbb{E}S^{2k}}{\mathbb{E}G^{2k}} = \sum_{k\geq 0} b_k \frac{x^k}{k!}$$

On the other hand

$$\mathbb{E}e^{\sqrt{2x}S} = \mathbb{E}\prod_{i=1}^{n} e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^{n}\mathbb{E}e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^{n}\cosh\left(\sqrt{2x}a_i\right)$$

Crucially

$$\cosh(z) = \prod_{l=1}^{\infty}\left(1 + \frac{4z^2}{\pi^2(2l-1)^2}\right)$$

$$\mathbb{E}e^{\sqrt{2x}S} = \sum_{k\geq 0} \frac{\sqrt{2x}^k}{k!}\mathbb{E}S^k = \sum_{k\geq 0} \frac{\sqrt{2x}^{2k}}{(2k)!}\mathbb{E}S^{2k}$$

$$= \sum_{k\geq 0} \frac{2^k x^k}{(2k-1)!!2^k k!}\mathbb{E}S^{2k} = \sum_{k\geq 0} \frac{x^k}{k!}\cdot\frac{\mathbb{E}S^{2k}}{\mathbb{E}G^{2k}} = \sum_{k\geq 0} b_k \frac{x^k}{k!}$$

On the other hand

$$\mathbb{E}e^{\sqrt{2x}S} = \mathbb{E}\prod_{i=1}^n e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^n \mathbb{E}e^{\sqrt{2x}a_i\varepsilon_i} = \prod_{i=1}^n \cosh\left(\sqrt{2x}a_i\right)$$

Crucially

$$\cosh(z) = \prod_{l=1}^{\infty}\left(1 + \frac{4z^2}{\pi^2(2l-1)^2}\right)$$

This gives

$$\mathbb{E}e^{\sqrt{2x}S} = \prod_{i=1}^n\prod_{l=1}^{\infty}\left(1 + \frac{8a_i^2}{\pi^2(2l-1)^2}x\right) = \prod_i (1 + c_i x).$$

$C_{p,q} = (\mathbb{E}|G|^p)^{1/p}/(\mathbb{E}|G|^q)^{1/q}$ (proof of N. and Tkocz)

$$\sum_{k\geq 0} b_k \frac{x^k}{k!} = \mathbb{E}e^{\sqrt{2x}S} = \prod_i (1 + c_i x) = \sum_{k\geq 0} \sigma_k x^k$$

$$\sum_{k\geq 0} b_k \frac{x^k}{k!} = \mathbb{E}e^{\sqrt{2x}S} = \prod_i (1 + c_i x) = \sum_{k\geq 0} \sigma_k x^k$$

Therefore

$$b_k = k!\sigma_k \qquad \text{is log-concave (by Newton)}$$

Thank you!